# LIMIT CANONICAL SYSTEMS ON CURVES WITH TWO COMPONENTS

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Abstract. In the 80's D. Eisenbud and J. Harris considered the following problem: "What are the limits of Weierstrass points in families of curves degenerating to stable curves?" But for the case of stable curves of compact type, treated by them, this problem remained wide open since then. In the present article, we propose a concrete approach to this problem, and give a quite explicit solution for stable curves with just two irreducible components meeting at points in general position.

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## 1. Introduction

**1.1.** *Goal.* In [EH2] D. Eisenbud and J. Harris asked: "What are the limits of Weierstrass points in families of curves degenerating to stable curves *not* of compact type?" As announced in [EM], in the present article we answer their question for stable curves with just two irreducible components meeting at points in general position.

For a glimpse of our results, consider the following particular case: Let $C$ be a nodal curve with two elliptic components $E_1$ and $E_2$ meeting at $\delta$ points $p_1, \dots, p_\delta$ in general position. View $C$ in $\mathbb{P}^{3\delta-1}$ under the bicanonical map, and let $H \subseteq \mathbb{P}^{3\delta-1}$ be the projective subspace spanned by $p_1, \dots, p_\delta$. For each hyperplane $L \subseteq H$ let $R_1(L)$ and $R_2(L)$ be the ramification divisors of the projections of $E_1$ and $E_2$ with center $L$, and put

$$W(L) := R_1(L) + R_2(L) + (\delta+1)(\delta-2)(p_1 + \cdots + p_\delta).$$

First author supported by PRONEX, Convênio 41/96/0883/00, CNPq, Proc. 300004/95-8, and FAPERJ, Proc. E-26/170.418/2000-APQ1.

It follows from our Theorem 10.7 (see Example 11.3 as well) that each $W(L)$ is limit of Weierstrass divisors in one–parameter families of smooth curves degenerating to $C$, and there are no other such limits.

**1.2.** *Some history.* Limits of ramification points and linear systems were studied by Eisenbud and Harris in the 80's, when they developed the theory of limit linear series for curves $C$ of compact type; see [EH1]. (As observed in [HM, p. 256], the theory works more generally when $C$ is *treelike*, that is, when the normalization of $C$ at all its irreducible nodes is of compact type.) Many important applications of their theory were found; a survey is given in [EH1].

Since many applications were obtained from a theory applicable only to curves of a special type, it's natural to expect more applications from a more general theory. In fact, Eisenbud and Harris wrote in [EH3, p. 220] that " ... there is probably a small gold mine awaiting a general insight." Despite the potential for applications and the vast interest the topic generated in the 80's, the theory of limit linear series could not be extended to stable curves of more general type. There was unpublished work of Z. Ran on degenerations of linear series [R], but the crucial relationship with degenerations of ramification points was yet to be established.

It was only recently that this relationship was established by the first author in [E2], building up on the work of Ran (or rather rediscovering it) and using the substitutes for the sheaves of (relative) principal parts discovered in [E1]. It is apparently possible to generalize the theory of limit linear series of [EH1] starting from results in [E2].

Though a more general theory of limit linear series is yet to be developed, it is already possible to apply the results in [E2] to answer the question raised by Eisenbud and Harris; see 1.1. For starters we describe in the present article a complete answer for stable curves with just two irreducible components meeting at points in general position. This is a substantial breakthrough, as the question was not completely answered even for the simplest case of stable curves with two elliptic components meeting at two points in general position.

In fact, it was only very recently that partial progress was made by M. Coppens and L. Gatto towards answering the question. In [CG] Coppens and Gatto consider a stable curve $C$ with just two irreducible components meeting at $\delta$ points in general position, and show that all the points in each collection of $\delta-1$ smooth points of $C$ are limits of Weierstrass points in a suitable family of smooth curves degenerating to $C$. (Their result is partially recovered by our Proposition 11.1.) Besides this general result, they study the case where one of the components of $C$ is elliptic and obtain partial results. In contrast with the methods used in the present article, they use admissible covers.

Besides [E2], the recent work by L. Mainò on enriched structures in her Ph.D. thesis [M] was important for us. Actually, as we deal here only with limits of canonical systems, we used only a fraction of what is available in [M].

There are further recent works on (generalizations of) limit linear series worth noticing, even though they are not completely related to what we do here. First, there is a draft by R. Pandharipande [P] who takes a Geometric Invariant Theory approach to proving the existence of what he calls generalized linear series for certain degenerations. Second, there is A. Bruno's thesis [B1], where he studies degenerations of linear systems to a stable curve with two components. It's particularly interesting the conditions he gives for such a curve to be a limit of smooth plane quintics. Finally, there is [B2], where applications to (variants of) a problem of Severi's are given.

We thank A. Bruno, M. Coppens, D. Eisenbud, J. Harris, S. Kleiman and specially L. Gatto and L. Mainò for helpful conversations.

**1.3.** *The general approach.* Rather than producing directly a space parameterizing limits of Weierstrass divisors on a stable curve, our approach consists of producing a variety $\mathbb{V}$ of limits of canonical systems and a formula that gives for each point of $\mathbb{V}$ the corresponding limit of Weierstrass divisors. This method is applicable to any stable curve, as we shall describe below.

Let $C$ be a connected, local complete intersection, projective curve of arithmetic genus $g > 0$ defined over an algebraically closed field $k$ of characteristic 0. Let $B := \mathrm{Spec}(k[[t]])$, and denote by $o$ its special point and $\eta$ its generic point. A *smoothing* of $C$ is a projective and flat map $\pi\colon S \to B$ such that $S_\eta$ is smooth and $S_o = C$. For each smoothing $\pi\colon S \to B$ of $C$ let $W_\pi := \overline{W}_\eta \cap C$, where $W_\eta \subseteq S$ is the Weierstrass subscheme of $S_\eta$. We call the associated cycle $[W_\pi]$ the *limit Weierstrass divisor of* $\pi$.

Let $C_1, \dots, C_n$ denote the irreducible components of $C$. A smoothing $\pi\colon S \to B$ of $C$ is called *regular* if $S$ is regular everywhere but possibly at the singularities of $C$ that lie on just one component $C_i$. If $\pi\colon S \to B$ is a regular smoothing of $C$, then $C_1, \dots, C_n$ are Cartier divisors on $S$.

Assume from now on that the irreducible components of $C$ meet at nodes, that is, ordinary double points. Let $\Delta$ denote the set of points of intersection between distinct components of $C$. For each $\mu$ in $\mathbb{Z}^+_\Delta := \prod_{p\in\Delta} \mathbb{Z}^+$ let $\widetilde{C}$ be the curve obtained from $C$ by splitting the branches of $C$ at each $p \in \Delta$ and connecting them by a chain of $\mu_p - 1$ rational, smooth curves (see Figure 1). We say that $\widetilde{C}$ is the *$\mu$–semi–stable model of* $C$. The smoothings of $C$

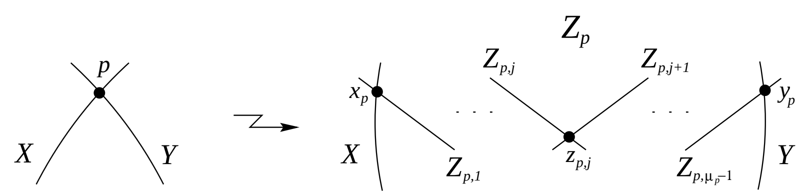


Figure 1. Semi–stable reduction.

correspond to the regular smoothings of the semi–stable models of $C$. This correspondence is called *semi–stable reduction*; see 2.7. We shall view the irreducible components of $C$ as irreducible components of $\widetilde{C}$.

For each $i = 1, \dots, n$ let $d_i \in \mathbb{Z}$ and put $\mathbb{L}_i := \omega_i((1 + d_i)\sum_{p\in\Delta_i} p)$, where $\omega_i$ is the dualizing sheaf of $C_i$ and $\Delta_i := \Delta \cap C_i$.

Let $\mu \in \mathbb{Z}^+_\Delta$ and consider a regular smoothing $\widetilde{\pi}\colon \widetilde{S} \to B$ of the $\mu$–semi–stable model $\widetilde{C}$ of $C$. Let $\omega_{\widetilde{\pi}}$ be the (relative) dualizing sheaf of $\widetilde{\pi}$. By Theorem 2.2, for each $i = 1, \dots, n$ there is a unique Cartier divisor $D_i$ on $\widetilde{S}$ that is supported in the union of all irreducible components of $\widetilde{C}$ but $C_i$ and such that the following two properties hold for $\mathcal{L}_i := \omega_{\widetilde{\pi}}(D_i)$.

1. The natural map $\varrho_i\colon H^0(\mathcal{L}_i)(o) \to H^0(\mathcal{L}_i|_{C_i})$ is injective.
2. The natural map $\varrho_{i,j}\colon H^0(\mathcal{L}_i)(o) \to H^0(\mathcal{L}_i|_{C_j})$ is not zero if $j \neq i$.

Let $V_{\widetilde{\pi},i} := \mathrm{Im}(\varrho_i)$. We call $\mathcal{L}_i$ the *canonical sheaf of* $\widetilde{\pi}$ *with focus on* $C_i$ and $(V_{\widetilde{\pi},i}, \mathcal{L}_i|_{C_i})$ the *limit canonical aspect of* $\widetilde{\pi}$ *with focus on* $C_i$. For each $p \in \Delta_i$ denote by $\ell_i(p)$ the coefficient in $D_i$ of the irreducible component of $\widetilde{C}$ other than $C_i$ containing $p$. We call $\ell_i(p)$ the *correction*

*number for $\mathcal{L}_i$ at $p$.* Let $d_{\widetilde{\pi},i} := \max\{\ell_i(p) \,|\, p \in \Delta_i\}$. If $d_i \geq d_{\widetilde{\pi},i}$ then $\mathcal{L}_i|_{C_i} \subseteq \mathbb{L}_i$, and thus $V_{\widetilde{\pi},i}$ can be viewed as a $g$–dimensional subspace of $H^0(\mathbb{L}_i)$.

Assume from now on that $d_i \geq d_{\widetilde{\pi},i}$ for every regular smoothing $\widetilde{\pi}$ of every semi–stable model of $C$ and every $i = 1, \dots, n$. (This condition holds if there are no loosely connected rational tails in $C$; see Remark 2.5.) Let

$$\mathbb{G} := \mathrm{Grass}_g(H^0(\mathbb{L}_1)) \times \cdots \times \mathrm{Grass}_g(H^0(\mathbb{L}_n)).$$

Given a regular smoothing $\widetilde{\pi}$ of a semi–stable model of $C$ let

$$\nu_{\widetilde{\pi}} := (V_{\widetilde{\pi},1}, \dots, V_{\widetilde{\pi},n}) \in \mathbb{G}.$$

For each $\mu \in \mathbb{Z}^+_\Delta$ let

$$\mathbb{V}_\mu := \{\nu_{\widetilde{\pi}} \,|\, \widetilde{\pi} \text{ is a regular smoothing of the } \mu\text{–semi–stable model of } C\}.$$

Put

$$\mathbb{V} := \bigcup_{\mu \in \mathbb{Z}^+_\Delta} \mathbb{V}_\mu \subseteq \mathbb{G}. \tag{1.3.1}$$

We call $\mathbb{V}$ the *variety of limit canonical systems of $C$*.

For each $\nu = (V_1, \dots, V_n) \in \mathbb{G}$ let

$$W_\nu := R_{\nu,1} + \cdots + R_{\nu,n} + \sum_{i<j} \sum_{p \in C_i \cap C_j} g(g-1-d_i-d_j)p,$$

where $R_{\nu,i}$ is the ramification divisor of $(V_i, \mathbb{L}_i)$ for each $i = 1, \dots, n$. By Theorem 2.8 (see Remark 2.9 as well), if $\widetilde{\pi}$ is a regular smoothing of a semi–stable model of $C$ then $W_{\nu_{\widetilde{\pi}}}$ is the limit Weierstrass divisor of the corresponding smoothing of $C$.

So, in order to answer the question posed by Eisenbud and Harris (see 1.1), it's enough to understand $\mathbb{V}$ well. We propose the following problems:

1. Describe explicitly each $\mathbb{V}_\mu$ and the interrelations among the various $\mathbb{V}_\mu$.
2. Find properties of $\mathbb{V}$. For instance, is $\mathbb{V}$ projective? Is $\mathbb{V}$ connected? What is the dimension of $\mathbb{V}$? When is $\mathbb{V}$ irreducible? What is the number of irreducible components of $\mathbb{V}$?

**1.4.** *Results.* Solving the problems raised at the end of 1.3 can be difficult without further hypotheses. In the present article we apply the approach of 1.3 for curves with just two irreducible components meeting at points in general position. We describe our results below.

Let $C$ be a connected, local complete intersection, projective curve of arithmetic genus $g > 0$ defined over an algebraically closed field $k$ of characteristic 0. Assume that $C$ has only two irreducible components, denoted $X$ and $Y$, and that the points of $X \cap Y$ are nodes of $C$. Let $\Delta := X \cap Y$ and $\delta := |\Delta|$. Let $g_X$ and $g_Y$ be the arithmetic genera of $X$ and $Y$. Then $g = g_X + g_Y + \delta - 1$. Assume that $\delta > 1$ or $g_X g_Y > 0$. Let $\omega_X$ and $\omega_Y$ denote the dualizing sheaves of $X$ and $Y$.

Assume from now on that $\Delta$ lies in general position on $X$ and $Y$. More precisely, assume that each effective divisor $D$ on $X$ (resp. $Y$) supported in $\Delta$ and of degree at most $g_X$ (resp. $g_Y$) imposes independent conditions on $\omega_X$ (resp. $\omega_Y$).

Let $\mathbb{V}$ be the variety of limit canonical systems of $C$, as described in 1.3. Our Theorem 10.6 asserts that $\mathbb{V}$ is connected, projective and of pure dimension. Moreover, $\dim \mathbb{V} = \delta - 1$ unless $g_X = g_Y = 0$, in which case $\mathbb{V}$ is a point. Our Theorem 11.2 gives a lower bound for the

number $N(\mathbb{V})$ of irreducible components of $\mathbb{V}$. Using this lower bound we show that $\mathbb{V}$ is irreducible if and only if $g_X, g_Y \leq 1$. If $\delta = 2$ then $N(\mathbb{V}) = g - \gcd(g_X + 1, g_Y + 1)$; see 11.5.

We solve Problem 1 of 1.3 as well. In fact, our solution to Problem 2 follows from our quite explicit solution to Problem 1. We proceed as follows. To each $\mu$ in $\mathbb{Z}^+_\Delta := \prod_{p\in\Delta} \mathbb{Z}^+$ we associate non–empty subsets $I_\mu, J_\mu \subseteq \Delta$ and $\delta$–tuples $\alpha_\mu, \beta_\mu \in \mathbb{Z}^+_\Delta$ such that $\alpha_{\mu,p} \leq g_Y$ and $\beta_{\mu,p} \leq g_X$ for each $p \in \Delta$; see 4.1. These associated data $\alpha_\mu, \beta_\mu, I_\mu, J_\mu$ are obtained numerically from $\mu$ alone, but they have geometric meaning as well. In fact, let $\widetilde{C}$ be the $\mu$–semi–stable model of $C$ and $\phi\colon \widetilde{C} \to C$ the induced map. Let $\mathcal{L}$ be the canonical sheaf with focus on $X$ (resp. $Y$) of a regular smoothing of $\widetilde{C}$. Let $p \in \Delta$. Then $p \in I_\mu$ (resp. $p \in J_\mu$) if and only if $\phi_*(\mathcal{L}|_{\widetilde{C}})$ is invertible at $p$; see Remark 5.3. In addition, $\alpha_{\mu,p}$ (resp. $\beta_{\mu,p}$) is the correction number for $\mathcal{L}$ at the point of $\widetilde{C}$ lying on $X$ (resp. $Y$) and above $p$; see Theorem 5.2. So $\mathbb{V}_\mu \subseteq \mathbb{G}$, where

$$\mathbb{G} := \mathrm{Grass}_g(H^0(\omega_X((1+g_Y)\textstyle\sum_{p\in\Delta} p))) \times \mathrm{Grass}_g(H^0(\omega_Y((1+g_X)\textstyle\sum_{p\in\Delta} p))).$$

It's interesting to note that, since $I_\mu$ and $J_\mu$ are non–empty, the canonical sheaves with foci on $X$ and $Y$ of every regular smoothing of every semi–stable model of $C$ push forward to (relatively) simple sheaves on the corresponding smoothing of $C$. (Recall that a sheaf $L$ on $C$ is *simple* if $\mathrm{End}(L) = k$.) So we need only consider sheaves that are simple on $C$ to obtain a projective variety $\mathbb{V}$. This is the same situation found in [E3], where a compactification of the (relative) Jacobian of a family of curves was constructed using only simple sheaves.

Theorem 10.1 asserts the importance of the data $\alpha_\mu, \beta_\mu, I_\mu, J_\mu$: they determine $\mathbb{V}_\mu$. In fact, an explicit description of $\mathbb{V}_\mu$ from these data is given by Theorem 8.2. As a corollary we obtain Theorem 8.5, which asserts that each $\mathbb{V}_\mu$ is isomorphic to a torus and gives a formula for the dimension of $\mathbb{V}_\mu$.

Theorem 10.1 asserts as well that any two $\mathbb{V}_\mu$ and $\mathbb{V}_{\mu'}$ are either equal or disjoint. Now, there are only finitely many $\mathbb{V}_\mu$ as there are only finitely many associated data $\alpha_\mu, \beta_\mu, I_\mu, J_\mu$. So the covering (1.3.1) can actually be thought of as a finite stratification of $\mathbb{V}$.

It remains to describe the relationship between the several $\mathbb{V}_\mu$. First, Theorem 10.1 says that $\mathbb{V}_{r\mu} = \mathbb{V}_\mu$ for each positive integer $r$. This fact is not surprising as each regular smoothing of the $\mu$–semi–stable model of $C$ induces a regular smoothing of the $r\mu$–semi–stable model after base change of degree $r$ and resolution of singularities. At any rate, this fact allows us to define $\mathbb{V}_\mu$ for each $\mu$ in $\mathbb{Q}^+_\Delta := \prod_{p\in\Delta} \mathbb{Q}^+$ in the obvious way. Then we obtain a remarkable relation between the Zariski topology of $\mathbb{V}$ and the Euclidean topology of $\mathbb{Q}^+_\Delta$: Theorem 10.5 asserts that there is a neighborhood $U_\mu \subseteq \mathbb{Q}^+_\Delta$ of each $\mu \in \mathbb{Q}^+_\Delta$ such that the closure $\overline{\mathbb{V}}_\mu \subseteq \mathbb{G}$ satisfies

$$\overline{\mathbb{V}}_\mu = \bigcup_{\overline{\mu}\in U} \mathbb{V}_{\overline{\mu}}$$

for every open neighborhood $U \subseteq U_\mu$ of $\mu$.

**1.5.** *An overview.* Let $C$ be a connected, local complete intersection, projective curve of arithmetic genus $g > 0$ defined over an algebraically closed field of characteristic 0. Assume that the irreducible components of $C$ meet at nodes.

In Section 2 we review the fundamentals of the theory of limit linear systems and limit ramification points on $C$ developed in [E2], as applied to limits of canonical systems and Weierstrass points. In contrast with [E2] we show here how to handle non–regular smoothings

of $C$ using semi–stable reduction; see 2.7 and Theorem 2.8. Semi–stable reduction can be incorporated to the general theory of limit linear systems and limit ramification points, but we refrain from carrying this out here.

In Section 3 we prove the purely numerical Lemma 3.1.

In Section 4 we define our basic set–up, introducing the terminology that shall be used in the rest of the article. Except in Section 6, it is assumed from here on that $C$ has just two irreducible components $X$ and $Y$. Let $\Delta := X \cap Y$. We state as well the general–position conditions, (4.3.1) and (4.3.2), that will be assumed later. Using Lemma 3.1, we define for each $\mu$ in $\mathbb{Z}^+_\Delta := \prod_{p\in\Delta}\mathbb{Z}^+$ the numerical data $\alpha_\mu, \beta_\mu, I_\mu, J_\mu$ which will a posteriori determine the stratum $\mathbb{V}_\mu$ of the variety of limit canonical systems $\mathbb{V}$.

In Section 5 we determine for each regular smoothing of a semi–stable model $\widetilde{C}$ of $C$ the associated canonical sheaves and limit canonical aspects with foci on $X$ and $Y$. Our main result here is Theorem 5.2.

In Section 6 we review the part of the theory of enriched structures of Mainò's which is necessary for us. Unfortunately, Mainò's results do not seem to meet directly our needs. So we follow an approach that is slightly different from hers, and give independent proofs. For instance, for us an enriched structure is a collection of invertible fractional ideals on $C$ satisfying certain properties, rather than just the collection of their associated (abstract) invertible sheaves; see 6.4 and Remark 6.7. The main result of this section, Theorem 6.5, gives a criterion for when a collection of fractional ideals on $C$ comes from an enriched structure. It would actually be more useful to have instead a similar criterion valid for collections of invertible sheaves.

In Section 7 we begin our study of the regeneration process. More precisely, we fix $\mu \in \mathbb{Z}^+_\Delta$, and determine which invertible sheaves on the $\mu$–semi–stable model $\widetilde{C}$ of $C$ are restrictions of canonical sheaves of regular smoothings of $\widetilde{C}$ with foci on $X$ and/or $Y$. Theorem 6.5 is our main tool.

In Section 8 we apply the theorem of Section 7 to give a characterization of the stratum $\mathbb{V}_\mu$ in terms of the numerical data $\alpha_\mu, \beta_\mu, I_\mu, J_\mu$. Our main results here are Theorems 8.2 and 8.5.

In Section 9 we study boundary points of certain tori orbits in products of Grassmannians. This section does not rely on the rest of the paper and might be of independent interest. In fact, Lemma 9.1 is probably well known. We give a proof of it as the proof of Lemma 9.2 is based on that proof.

In Section 10 we prove our most important results, Theorems 10.1, 10.5, 10.6 and 10.7. Here we tackle all strata $\mathbb{V}_\mu$ of $\mathbb{V}$ together and determine which lie in the closure in $\mathbb{V}$ of which. Lemma 9.2 is needed for this purpose.

In Section 11 we collect a few results: Proposition 11.1 recovers in part the main result of [CG] and Theorem 11.2 gives estimates on the number of irreducible components of $\mathbb{V}$. In addition, we describe how to represent $\mathbb{V}$ graphically (see 11.4) and describe this representation for $|\Delta| = 2$ and $|\Delta| = 3$; see 11.5–6.

**1.6.** *Notation.* If $F$ and $G$ are sets, let $G_F := \prod_{p\in F} G$. If $G$ is a group then so is $G_F$; in addition, if $E \subseteq F$ we shall view $G_E$ as a subgroup of $G_F$ in the natural way.

Let $F$ be a finite set. Given $b, c \in \mathbb{Z}_F$ write $b \leq c$ if $b_p \leq c_p$ for every $p \in F$. For each $b \in \mathbb{Z}_F$ let $|b| := \sum_{p\in F} b_p$. View each $p \in F$ as an element of $\mathbb{Z}_F$, defined by $p_p := 1$ and $p_q := 0$ if $q \neq p$. In addition, view each $b \subseteq F$ as the sum $\sum_{p\in b} p$ inside $\mathbb{Z}_F$.

Let $k$ be a field. Given $b \in \mathbb{Z}_F$ and $t \in k_F^*$, let $t^b := \prod_{p\in F} t_p^{b_p}$. The group $k_F^*$ acts diagonally on $k_F$: given $t \in k_F^*$ and $v \in k_F$, define $t \cdot v \in k_F$ by $(t \cdot v)_p := t_p v_p$ for each $p \in F$.

If $\mathcal{F}$ is a coherent sheaf and $D$ is a Cartier divisor on a scheme $X$, let $\mathcal{O}_X(D)$ denote the invertible sheaf associated to $D$, and put $\mathcal{F}(D) := \mathcal{F} \otimes \mathcal{O}_X(D)$. If $X$ is a scheme and $Y$ a closed subscheme, let $[Y]$ denote the associated cycle on $X$.

In the present article, a *curve* is a connected, reduced, *local complete intersection*, projective scheme of pure dimension 1 defined over an algebraically closed field of characteristic 0. A singularity of a curve is called *irreducible* if it lies on just one irreducible component, and *reducible* if not. It is called a *node* if it is an ordinary double point. In the present article, a curve is called *nodal* if all its *reducible* singularities are nodes. In addition, a curve is called *treelike* if its normalization at all irreducible singularities is of compact type.

## 2. Limit canonical systems and Weierstrass divisors

**2.1.** *Limit canonical systems.* Let $C$ be a curve and $C_1, \dots, C_n$ its irreducible components. A *smoothing* of $C$ is a projective and flat map $\pi\colon S \to B$, where $B$ is the spectrum of a discrete valuation ring, the special fiber of $\pi$ is $C$ and the generic fiber is smooth. If $\pi$ is a smoothing of $C$ let $\omega_\pi$ denote the (relative) dualizing sheaf of $\pi$. Then $\omega_\pi$ is an invertible extension to $S$ of the canonical sheaf on the generic fiber of $\pi$.

A smoothing $\pi\colon S \to B$ of $C$ is said to be *regular* if $S$ is regular everywhere *but* possibly at the irreducible singularities of $C$.

Let $\pi\colon S \to B$ be a regular smoothing of $C$ and $o$ the special point of $B$. Then $C_i$ is a Cartier divisor on $S$ for each $i = 1, \dots, n$. (In fact, $C_i$ is principal away from the irreducible singularities of $C$ because $S$ is regular there, and at each irreducible singularity of $C$ in $C_i$ because $C_i$ is locally the pull–back of $o$.) In addition, $C_1 + \dots + C_n$ is the pull–back of $o$, hence linearly equivalent to 0. Since $C$ is connected, a $\mathbb{Z}$–linear combination of $C_1, \dots, C_n$ is linearly equivalent to 0 if and only if it is a multiple of $C_1 + \dots + C_n$.

If $\mathcal{L}$ is an invertible extension to $S$ of the canonical sheaf on the generic fiber of $\pi$, then

$$\mathcal{L} \cong \omega_\pi(t_1C_1 + \dots + t_nC_n)$$

for certain integers $t_1, \dots, t_n$. We say that $\mathcal{L}$ is a *canonical sheaf of* $\pi$. In particular, $\omega_\pi$ is a canonical sheaf.

For each canonical sheaf $\mathcal{L}$ of $\pi$, the base–change map $\varrho_{\mathcal{L}}\colon H^0(\mathcal{L})(o) \to H^0(\mathcal{L}|_C)$ is injective. We say that $(\mathrm{Im}(\varrho_{\mathcal{L}}), \mathcal{L}|_C)$ is a *limit canonical system of* $\pi$.

**Theorem 2.2.** *Let $C$ be a nodal curve of arithmetic genus $g > 0$ and $C_1, \dots, C_n$ its irreducible components. Let $\pi\colon S \to B$ be a regular smoothing of $C$ and $o$ the special point of $B$. Then for each $i = 1, \dots, n$ there is a unique canonical sheaf $\mathcal{L}_i$ of $\pi$ meeting the following two conditions.*

1. *The natural map $\varrho_i\colon H^0(\mathcal{L}_i)(o) \to H^0(\mathcal{L}_i|_{C_i})$ is injective.*
2. *The natural map $\varrho_{i,j}\colon H^0(\mathcal{L}_i)(o) \to H^0(\mathcal{L}_i|_{C_j})$ is not zero if $j \neq i$.*

*Proof.* As in [E2, Thm. 1, p. 23 and Prop. 2, p. 26]. See [R] as well. □

**Definition 2.3.** Keep the set–up of Theorem 2.2. For each $i = 1, \dots, n$ we say that $\mathcal{L}_i$ is the *canonical sheaf of $\pi$ with focus on $C_i$*. The associated limit canonical system is also said to have focus on $C_i$. In addition, we call the linear system $(\mathrm{Im}(\varrho_i), \mathcal{L}_i|_{C_i})$ the *limit canonical aspect of $\pi$ with focus on $C_i$*.

For each $i=1,\dots,n$ there are unique integers $t_{i,1},\dots,t_{i,n}$ such that $t_{i,i}=0$ and

$$\mathcal{L}_i \cong \omega_\pi(\sum_{j=1}^n t_{i,j}C_j).$$

For each reducible node $p$ of $C$ in $C_i$ let $\ell_i(p) := t_{i,j}$, where $j$ is the unique integer such that $j \neq i$ and $p \in C_j$. Call $\ell_i(p)$ the *correction number for $\mathcal{L}_i$ at $p$.*

**Proposition 2.4.** *Let $C$ be a nodal curve of arithmetic genus $g > 0$ and $C_1,\dots,C_n$ its irreducible components. Let $\pi\colon S \to B$ be a regular smoothing of $C$. If $\mathcal{M}$ is a canonical sheaf of $\pi$ such that the restriction $H^0(\mathcal{M}) \to H^0(\mathcal{M}|_{C_i})$ is not zero for every $i=1,\dots,n$, then for each $i=1,\dots,n$ there are unique non–negative integers $t_{i,1},\dots,t_{i,n}$ such that $t_{i,i}=0$ and*

$$\mathcal{L}_i \cong \mathcal{M}(\sum_{j=1}^n t_{i,j}C_j),$$

*where $\mathcal{L}_i$ is the canonical sheaf of $\pi$ with focus on $C_i$.*

*Proof.* As in [E2, Prop. 4, p. 27]. □

**Remark 2.5.** Keep the set–up of Proposition 2.4. For each $i=1,\dots,n$ let $e_i$ be the minimum non–negative integer such that the restriction,

$$H^0(\omega_\pi(-e_iC_i)) \longrightarrow H^0(\omega_\pi(-e_iC_i)|_{C_i}),$$

is not zero. Since forming $\omega_\pi$ commutes with changing $B$, we have $e_i > 0$ if and only if $C_i$ is smooth and rational and there are as many connected components of $\overline{C - C_i}$ as points in $C_i \cap \overline{C - C_i}$. (In this case, $C_i$ is called a *loosely connected rational tail*; see [C, Def. 3.2, p. 75].)

Let

$$\mathcal{M} := \omega_\pi(-\sum_{i=1}^n e_iC_i).$$

By Proposition 2.4, for each $i=1,\dots,n$, there are non–negative integers $t_{i,1},\dots,t_{i,n}$ such that $t_{i,i}=0$ and

$$\mathcal{L}_i \cong \mathcal{M}(\sum_{j=1}^n t_{i,j}C_j).$$

If $p \in C_i \cap C_j$ for $j \neq i$ then $t_{i,j} \leq 2g-2$ because $\deg \mathcal{L}_i|_{C_i} \leq 2g-2$. Observe that $\ell_i(p) = t_{i,j} - e_j + e_i$. Thus $\ell_i(p) \leq 2g-2$ if $C_i$ is not a loosely connected rational tail.

**Definition 2.6.** Let $\pi\colon S \to B$ be a smoothing of a curve $C$. The Weierstrass scheme on the generic fiber of $\pi$ extends to a unique closed subscheme $\mathcal{W} \subseteq S$ that is flat over $B$. We say that the special fiber $W$ of $\mathcal{W}$ is the *limit Weierstrass scheme of* $\pi$, and call the associated cycle $[W]$ the *limit Weierstrass divisor of* $\pi$.

**2.7.** *Semi–stable reduction.* Let $C$ be a nodal curve and $\pi\colon S \to B$ a smoothing of $C$. If $p$ is a node of $C$, then the local equation for $\pi$ at $p$ is of the form $t^{\mu_p} = h_1h_2$, where $\mu_p$ is a positive integer, $t$ is a local parameter of $B$ at the special point, and $h_1$ and $h_2$ are the local

equations of the branches of $C$ at $p$. The integer $\mu_p$ is called the *singularity type of* $\pi$ *at* $p$. The smoothing is regular if and only if its singularity type at every reducible node of $C$ is 1.

If $S$ is not regular, then we resolve its singularities at the reducible nodes of $C$ by blowing up successively. We obtain a regular smoothing $\widetilde{\pi}\colon \widetilde{S} \to B$ whose generic fiber is equal to that of $\pi$, and whose special fiber $\widetilde{C}$ is the curve obtained from $C$ by splitting the branches of $C$ at each reducible node $p$, and connecting them by a chain $Z_p$ of $\mu_p - 1$ rational, smooth curves $Z_{p,1}, \dots, Z_{p,\mu_p-1}$, where $\mu_p$ is the singularity type of $\pi$ at $p$ (see Figure 1). We say that the curve $\widetilde{C}$ described above is a *semi–stable model of* $C$, and call $\widetilde{\pi}$ the *semi–stable reduction of* $\pi$. We'll view the irreducible components of $C$ as irreducible components of $\widetilde{C}$.

Conversely, let $\widetilde{\pi}'\colon \widetilde{S}' \to B'$ be any regular smoothing of the curve $\widetilde{C}$ described above. Blow down the chains $Z_p$ of rational, smooth curves, and let $S'$ denote the ensuing surface. Since the only curves contracted lie on the special fiber of $\widetilde{\pi}$, the map $\widetilde{\pi}'$ descends to a map $\pi'\colon S' \to B'$. The self–intersection of each $Z_{p,j}$ in $\widetilde{S}'$ is $-2$ because $\widetilde{S}'$ is regular along $Z_{p,j}$. So, each chain $Z_p$ is blown down to a node, and we recover the curve $C$ as the special fiber of $\pi'$. Thus $\pi'$ is a smoothing of $C$.

Let $\phi\colon \widetilde{S} \to S$ denote the birational map. Let $\omega$ and $\widetilde{\omega}$ denote the (relative) dualizing sheaves of $\pi$ and $\widetilde{\pi}$, respectively. Since $C$ is Gorenstein, $\omega$ and $\widetilde{\omega}$ are invertible. Since $\widetilde{\omega}$ and $\phi^*\omega$ coincide on the generic fiber of $\widetilde{\pi}$, and on every irreducible component of $\widetilde{C}$, the two sheaves are equal. Hence, by the projection formula, $\omega = \phi_*\widetilde{\omega}$. So, if $W$ and $\widetilde{W}$ are the limit Weierstrass schemes of $\pi$ and $\widetilde{\pi}$ then $[W] = \phi_*[\widetilde{W}]$.

Therefore, to determine all limit Weierstrass divisors on $C$ we need only determine the limit Weierstrass divisor of each regular smoothing of each semi–stable model of $C$.

**Theorem 2.8.** *Let $C$ be a nodal curve of arithmetic genus $g > 0$ and $C_1, \dots, C_n$ its irreducible components. Let $N$ be the set of reducible nodes of $C$. Let $\pi$ be a smoothing of $C$ and $\widetilde{\pi}$ its semi–stable reduction. Let $\widetilde{C}$ be the special fiber of $\widetilde{\pi}$. For each $i = 1, \dots, n$, let $R_i$ be the ramification divisor of the limit canonical aspect of $\widetilde{\pi}$ with focus on $C_i$. For each $i = 1, \dots, n$ and each $p \in N \cap C_i$ let $\ell_i(p)$ be the correction number for the canonical sheaf of $\widetilde{\pi}$ with focus on $C_i$ at the point of $\widetilde{C}$ in $C_i$ lying above $p$. Then the limit Weierstrass scheme $W$ of $\pi$ satisfies*

$$[W] = \sum_{i=1}^{n} R_i + \sum_{i<j} \sum_{p \in C_i \cap C_j} g(g - 1 - \ell_i(p) - \ell_j(p))p.$$

*Proof.* Let $p \in N$, say $p \in C_i \cap C_j$ where $1 \le i < j \le n$. Let $\mu_p$ be the singularity type of $\pi$ at $p$. Let $Z_{p,1}, \dots, Z_{p,\mu_p-1}$ be the chain of rational, smooth curves on $\widetilde{C}$ that are contracted to $p$ in $C$. Assume that $Z_{p,1}$ intersects $C_i$ and $Z_{p,\mu_p-1}$ intersects $C_j$. Put $Z_{p,0} := C_i$ and $Z_{p,\mu_p} := C_j$. For each $h = 0, \dots, \mu_p - 1$ let $z_{p,h}$ be the point of intersection of $Z_{p,h}$ and $Z_{p,h+1}$ (see Figure 1).

For each $h = 0, \dots, \mu_p$ let $\mathcal{N}_{p,h}$ denote the canonical sheaf of $\widetilde{\pi}$ with focus on $Z_{p,h}$. For each $h = 1, \dots, \mu_p - 1$ let $\lambda_{p,h}$ and $\nu_{p,h}$ be the correction numbers for $\mathcal{N}_{p,h}$ at $z_{p,h-1}$ and $z_{p,h}$, respectively. Since $Z_{p,h}$ is smooth and rational, and its self–intersection is $-2$, it follows that $\mathcal{N}_{p,h}|_{Z_{p,h}}$ has degree $\lambda_{p,h} + \nu_{p,h}$. Hence, by Plücker's formula, the ramification divisor $R_{p,h}$ of the limit canonical aspect of $\widetilde{\pi}$ with focus on $Z_{p,h}$ satisfies

$$\deg R_{p,h} = g(\lambda_{p,h} + \nu_{p,h} - (g - 1)). \tag{2.8.1}$$

Put $\nu_{p,0} := \ell_i(p)$ and $\lambda_{p,\mu_p} := \ell_j(p)$. Then

$$\mathcal{N}_{p,h+1} \cong \mathcal{N}_{p,h}((\lambda_{p,h+1}+\nu_{p,h})Z_{p,h}+0Z_{p,h+1}+\cdots)$$

for $h=0,\dots,\mu_p-1$.

By [E2, Thm. 7, p. 30] the limit Weierstrass scheme $\widetilde{W}$ of $\widetilde{\pi}$ satisfies

$$[\widetilde{W}] = \sum_{i=1}^{n} R_i + \sum_{p\in N}\Big(\sum_{h=1}^{\mu_p-1} R_{p,h} + \sum_{h=0}^{\mu_p-1} g(g-1-\lambda_{p,h+1}-\nu_{p,h})z_{p,h}\Big). \tag{2.8.2}$$

Now, $[W]$ is the push–forward of $[\widetilde{W}]$ to $C$. Combining (2.8.1) and (2.8.2) we get the expression for $[W]$ claimed. □

**Remark 2.9.** Keep the set–up of Theorem 2.8. For each $i=1,\dots,n$ let $\omega_i$ be the dualizing sheaf of $C_i$. Assume that for each $i=1,\dots,n$ there is an integer $d_i$ such that $\ell_i(p)\leq d_i$ for each $p\in N\cap C_i$. (For instance, we may let $d_i := 2g-2$ for each $i=1,\dots,n$ if no irreducible component of $C$ is a loosely connected rational tail; see Remark 2.5.) Then, instead of viewing the limit canonical aspect of $\widetilde{\pi}$ with focus on $C_i$ as a system with sections in $\omega_i(\sum_{p\in N\cap C_i}(1+\ell_i(p))p)$, we may view it as having sections in $\omega_i((1+d_i)\sum_{p\in N\cap C_i}p)$. Let $R_i(d_i)$ be the corresponding ramification divisor. Then

$$[W] = \sum_{i=1}^{n} R_i(d_i) + \sum_{i<j} g(g-1-d_i-d_j)\sum_{p\in C_i\cap C_j} p.$$

## 3. A numerical lemma

**Lemma 3.1.** *Let $\Delta$ be a non–empty finite set and $\mu\in\mathbb{Q}_\Delta^+$. For each $\upsilon\in\mathbb{Z}$ there are unique $\alpha\in\mathbb{Z}_\Delta$ and $\rho\in\mathbb{Q}_\Delta$ satisfying the following four conditions.*

(3.1.1a) $\quad 0<\rho_p\leq\mu_p$ *for every* $p\in\Delta$,

(3.1.1b) $\quad I := \{p\in\Delta \mid \rho_p=\mu_p\}$ *is non–empty,*

(3.1.1c) $\quad \upsilon\leq|\alpha|<\upsilon+|I|$,

(3.1.1d) $\quad \mu_p(\alpha_p+1)-\rho_p = \mu_q(\alpha_q+1)-\rho_q$ *for all* $p,q\in\Delta$.

*If $\upsilon\geq 1-|\Delta|$ then $\alpha\geq 0$. If $\alpha\geq 0$ but $\alpha_p=0$ for some $p\in I$ then $\alpha=0$ and $I=\Delta$.*

*Proof.* The existence of $\alpha\in\mathbb{Z}_\Delta$ and $\rho\in\mathbb{Q}_\Delta$ satisfying Conditions (3.1.1) is equivalent to the existence of $\beta\in\mathbb{Z}_\Delta$ and $\sigma\in\mathbb{Q}_\Delta$ satisfying the following four conditions.

(3.1.2a) $\quad 0\leq\sigma_p\leq\mu_p$ for every $p\in\Delta$,

(3.1.2b) $\quad J := \{p\in\Delta \mid \sigma_p=0\}$ is non–empty,

(3.1.2c) $\quad |\beta|=\upsilon$,

(3.1.2d) $\quad \mu_p\beta_p+\sigma_p=\mu_q\beta_q+\sigma_q$ for all $p,q\in\Delta$.

In fact, suppose that $\beta\in\mathbb{Z}_\Delta$ and $\sigma\in\mathbb{Q}_\Delta$ satisfy Conditions (3.1.2). Define $\alpha\in\mathbb{Z}_\Delta$ and $\rho\in\mathbb{Q}_\Delta$ by letting for each $p\in\Delta$,

$$(\alpha_p,\rho_p) := \begin{cases}(\beta_p+1,\mu_p), & \text{if } \sigma_p=\mu_p,\\ (\beta_p,\mu_p-\sigma_p), & \text{otherwise.}\end{cases}$$

Then $\alpha$ and $\rho$ satisfy Conditions (3.1.1). Conversely, suppose that $\alpha \in \mathbb{Z}_\Delta$ and $\rho \in \mathbb{Q}_\Delta$ satisfy Conditions (3.1.1). By (3.1.1c), there is a proper subset $K$ of $I$ with $|K| = |\alpha| - \upsilon$. Define $\beta \in \mathbb{Z}_\Delta$ and $\sigma \in \mathbb{Q}_\Delta$ by letting for each $p \in \Delta$,

$$(\beta_p, \sigma_p) := \begin{cases} (\alpha_p - 1, \mu_p), & \text{if } p \in K, \\ (\alpha_p, \mu_p - \rho_p), & \text{otherwise.} \end{cases}$$

Then $\beta$ and $\sigma$ satisfy Conditions (3.1.2).

Let's prove now the existence of $\beta \in \mathbb{Z}_\Delta$ and $\sigma \in \mathbb{Q}_\Delta$ satisfying Conditions (3.1.2). Let $\Gamma$ be the set of points $x \in \mathbb{R}_\Delta$ such that

$$x_p = \mu_p\beta_p + \sigma_p \quad \text{for each } p \in \Delta, \tag{3.1.3}$$

where $\beta \in \mathbb{Z}_\Delta$ and $\sigma \in \mathbb{R}_\Delta$ satisfy Conditions (3.1.2a–c). Let

$$D := \{y \in \mathbb{R}_\Delta \mid y_p = y_q \text{ for all } p, q \in \Delta\}.$$

If $\Gamma$ meets $D$, then there are $\beta \in \mathbb{Z}_\Delta$ and $\sigma \in \mathbb{R}_\Delta$ satisfying Conditions (3.1.2). In addition, $\sigma \in \mathbb{Q}_\Delta$ by (3.1.2b) and (3.1.2d). Thus, it is enough to show that $\Gamma$ meets $D$.

If $x \in \Gamma$ then $\upsilon \leq \sum_{p\in\Delta} x_p/\mu_p < \upsilon + \delta$ by (3.1.2a–c) and (3.1.3). Since $D \not\subseteq P$, where

$$P := \{y \in \mathbb{R}_\Delta \mid \textstyle\sum_{p\in\Delta} y_p/\mu_p = 0\},$$

there is a point $x \in \Gamma$ closest to $D$. Fix this point $x$, and let $\beta \in \mathbb{Z}_\Delta$ and $\sigma \in \mathbb{R}_\Delta$ satisfying (3.1.2a–c) and (3.1.3). We'll show that $x \in D$.

Suppose $x \notin D$. Let $d\colon \mathbb{R}_\Delta \to \mathbb{R}$ be the distance function to $D$. Since the coordinate axes are not parallel to $D$ and $x \notin D$, we have $\partial_p d(x) \neq 0$ for every $p \in \Delta$. If $0 < \sigma_p < \mu_p$ for a certain $p \in \Delta$, then we would be able to produce a point of $\Gamma$ closer to $D$ by making $\sigma_p$ vary. So, either $\sigma_p = 0$ or $\sigma_p = \mu_p$ for each $p \in \Delta$. In addition, $\partial_p d(x) > 0$ if $\sigma_p = 0$ and $\partial_p d(x) < 0$ if $\sigma_p = \mu_p$.

Suppose that $\sigma_p = 0$ and $\sigma_q = \mu_q$ for certain $p, q \in \Delta$. Define $\beta' \in \mathbb{Z}_\Delta$ and $\sigma' \in \mathbb{Q}_\Delta$ by letting for each $r \in \Delta$,

$$(\beta'_r, \sigma'_r) := \begin{cases} (\beta_p - 1, \mu_p), & \text{if } r = p, \\ (\beta_q + 1, 0), & \text{if } r = q, \\ (\beta_r, \sigma_r), & \text{otherwise.} \end{cases}$$

Then $x_r = \mu_r\beta'_r + \sigma'_r$ for every $r \in \Delta$. However, since $\partial_q d(x) < 0$ and $\sigma'_q = 0$, we would get a point of $\Gamma$ closer to $D$ by increasing $\sigma'_q$, reaching a contradiction. Therefore, either $\sigma = 0$ or $\sigma = \mu$. By (3.1.2b), $\sigma = 0$, and thus $\partial_p d(x) > 0$ for every $p \in \Delta$.

Now, let $y \in D$ be the closest point to $x$, and $z := x - y$. Then $\sum_{p\in\Delta} z_p = 0$. But $z_p \geq 0$ for every $p \in \Delta$ because $\partial_p d(x) > 0$. Hence $z = 0$ and $x \in D$, reaching a contradiction. Thus $x \in D$, and hence $\beta$ and $\sigma$ satisfy Conditions (3.1.2).

Let's prove now uniqueness of $\alpha \in \mathbb{Z}_\Delta$ and $\rho \in \mathbb{Q}_\Delta$ meeting Conditions (3.1.1). Let $\alpha' \in \mathbb{Z}_\Delta$ and $\rho' \in \mathbb{Q}_\Delta$ satisfying the same conditions as $\alpha$ and $\rho$. Suppose $\alpha_p > \alpha'_p$ for some $p \in \Delta$. So,

$$\mu_q(\alpha'_q + 1) - \rho'_q = \mu_p(\alpha'_p + 1) - \rho'_p < \mu_p(\alpha_p + 1) - \rho_p = \mu_q(\alpha_q + 1) - \rho_q, \tag{3.1.4}$$

and hence $\alpha_q \geq \alpha'_q$ for every $q \in \Delta$, with equality only if $\rho'_q > \rho_q$. Thus $\alpha_q > \alpha'_q$ for every $q \in I$. Then

$$|\alpha| \geq |\alpha'| + |I| \geq \upsilon + |I|,$$

reaching a contradiction with (3.1.1c). So $\alpha \leq \alpha'$. Interchanging the roles of $\alpha$ and $\alpha'$, we get $\alpha = \alpha'$.

Suppose now that $\rho'_p > \rho_p$ for some $p \in \Delta$. So (3.1.4) holds, and hence $\rho'_q > \rho_q$ for every $q \in \Delta$. Since $\rho_q = \mu_q$ and $\mu_q \geq \rho'_q$ for every $q \in I$, we reach a contradiction. So, $\rho'_p \leq \rho_p$ for every $p \in \Delta$. Interchanging the roles of $\rho$ and $\rho'$, we get $\rho = \rho'$.

Finally, if $\alpha_p < 0$ for some $p \in \Delta$, then $\alpha_q < 0$ for every $q \in \Delta$ by Condition (3.1.1d). If so, $\upsilon \leq -|\Delta|$ by Condition (3.1.1c). Now, if $\alpha \geq 0$ but $\alpha_p = 0$ for some $p \in I$, then $\alpha = 0$ and $I = \Delta$ by Condition (3.1.1d). $\square$

**Definition 3.2.** Keep the set–up of Lemma 3.1. We call $(\alpha, \rho, I)$, and sometimes only $(\alpha, I)$, the *numerical data associated to $\mu$ and $\upsilon$.*

## 4. Set–up

**4.1.** *Two–component curves.* Let $C$ be a nodal curve with only two irreducible components, denoted $X$ and $Y$. Let $\Delta := X \cap Y$ and $\delta := |\Delta|$. To avoid exceptional cases, we shall assume that $\delta > 1$ or $g_X g_Y > 0$. (If also the irreducible singularities of $C$ are nodes, then we are assuming that $C$ is semi–stable.)

For each $p \in \Delta$, let $x_p$ and $y_p$ denote the points of $X$ and $Y$ lying over $p$. Let $g_X$ and $g_Y$ be the arithmetic genera of $X$ and $Y$. Then the arithmetic genus $g$ of $C$ is given by

$$g = g_X + g_Y + \delta - 1.$$

Let $\omega_X$ and $\omega_Y$ be the dualizing sheaves of $X$ and $Y$, and $\omega$ that of $C$. Let

$$\mathbb{L} := \omega_X((1+g_Y)\sum_{p\in\Delta} x_p) \quad \text{and} \quad \mathbb{M} := \omega_Y((1+g_X)\sum_{p\in\Delta} y_p).$$

Let $\mathbb{G}_X := \mathrm{Grass}_g(H^0(\mathbb{L}))$ and $\mathbb{G}_Y := \mathrm{Grass}_g(H^0(\mathbb{M}))$ denote the Grassmannians of $g$–dimensional vector subspaces of $H^0(\mathbb{L})$ and $H^0(\mathbb{M})$. Set $\mathbb{G} := \mathbb{G}_X \times \mathbb{G}_Y$.

For each $\mu \in \mathbb{Q}^+_\Delta$, let $(\alpha_\mu, \rho_\mu, I_\mu)$ be the numerical data associated with $\mu$ and $g_Y$, and $(\beta_\mu, \sigma'_\mu, J_\mu)$ the numerical data associated with $\mu$ and $g_X$; see Definition 3.2. Set $\sigma_\mu := \mu - \sigma'_\mu$.

Note that $\rho_\mu, \sigma_\mu \in \mathbb{Z}_\Delta$ if $\mu \in \mathbb{Z}^+_\Delta$. In this case, let $\gamma_\mu := \alpha_{\mu,p}\mu_p$ for (any) $p \in I_\mu$ and $\epsilon_\mu := \beta_{\mu,p}\mu_p$ for (any) $p \in J_\mu$. If $g_Y > 0$ then $\gamma_\mu > 0$ by Lemma 3.1. Analogously, if $g_X > 0$ then $\epsilon_\mu > 0$.

If $g_X g_Y > 0$ let $t \in \mathbb{Z}^+$ be such that $t\mu \in \mathbb{Z}^+_\Delta$ and define $\widetilde{\alpha}_\mu := \gamma_{t\mu}/\gcd(\gamma_{t\mu}, \epsilon_{t\mu})$ and $\widetilde{\beta}_\mu := \epsilon_{t\mu}/\gcd(\gamma_{t\mu}, \epsilon_{t\mu})$. Note that $\widetilde{\alpha}_\mu$ and $\widetilde{\beta}_\mu$ do not depend on the choice of $t$. In addition, if $p \in I_\mu \cap J_\mu$ then $\widetilde{\alpha}_\mu = \alpha_{\mu,p}/\gcd(\alpha_{\mu,p}, \beta_{\mu,p})$ and $\widetilde{\beta}_\mu = \beta_{\mu,p}/\gcd(\alpha_{\mu,p}, \beta_{\mu,p})$.

**4.2.** *Semi–stable reduction.* Preserve 4.1. Fix $\mu \in \mathbb{Z}^+_\Delta$. Abbreviate $\alpha_\mu$, $\rho_\mu$, $I_\mu$, $\beta_\mu$, $\sigma_\mu$, $J_\mu$, $\gamma_\mu$, $\epsilon_\mu$, $\widetilde{\alpha}_\mu$ and $\widetilde{\beta}_\mu$ by $\alpha$, $\rho$, $I$, $\beta$, $\sigma$, $J$, $\gamma$, $\epsilon$, $\widetilde{\alpha}$ and $\widetilde{\beta}$. Let

$$\begin{cases} L_X := \omega_X(\sum_{p\in\Delta}(1+\alpha_p)x_p), \\ L_Y := \omega_Y(\sum_{p\in I} y_p - \sum_{p\in\Delta}\alpha_p y_p), \\ M_Y := \omega_Y(\sum_{p\in\Delta}(1+\beta_p)y_p), \\ M_X := \omega_X(\sum_{p\in J} x_p - \sum_{p\in\Delta}\beta_p x_p). \end{cases} \tag{4.2.1}$$

By Lemma 3.1, $\alpha_p \geq 0$ for every $p \in \Delta$. In addition, $\alpha_p \leq g_Y$ for every $p \in \Delta$. Indeed, if $g_Y = 0$ then $\alpha = 0$. Now, if $g_Y > 0$ then $\alpha_p > 0$ for every $p \in I$; hence $\alpha_p \leq g_Y$ for every

$p \in \Delta$ because $|\alpha| < g_Y + |I|$. Likewise, $0 \leq \beta_p \leq g_X$ for every $p \in \Delta$. So we may (and eventually will) view $L_X$ and $M_Y$ as subsheaves of $\mathbb{L}$ and $\mathbb{M}$, respectively.

Denote by $\widetilde{C}$ the curve obtained from $C$ by splitting its branches at each $p \in \Delta$ and connecting them by a chain,

$$Z_p := Z_{p,1} \cup \cdots \cup Z_{p,\mu_p-1},$$

of $\mu_p - 1$ rational, smooth curves $Z_{p,j}$, as depicted in Figures 1 and 2. By convention, the

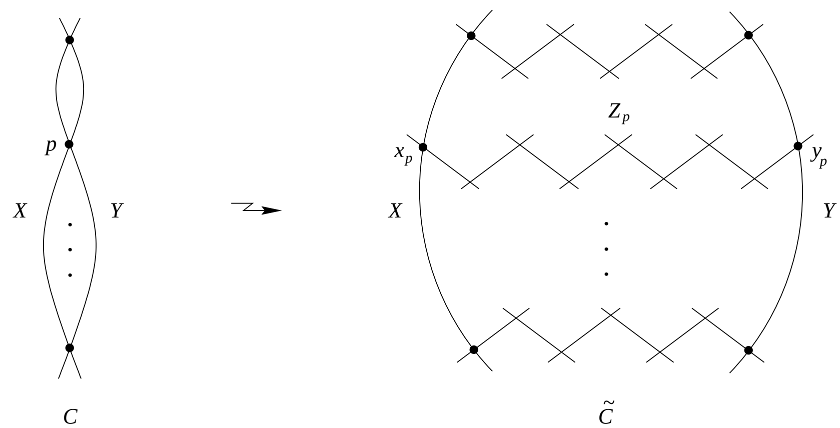


FIGURE 2. The curve $\widetilde{C}$.

leftmost curve in each chain $Z_p$ is $Z_{p,1}$ and the rightmost is $Z_{p,\mu_p-1}$. In addition, set $Z_{p,0} := X$ and $Z_{p,\mu_p} := Y$ for each $p \in \Delta$. For each $p \in \Delta$ and each $j = 0, \dots, \mu_{p-1}$, let $z_{p,j}$ be the point of $Z_{p,j} \cap Z_{p,j+1}$, as depicted in Figure 1. So $z_{p,0} = x_p$ and $z_{p,\mu_p-1} = y_p$ for every $p \in \Delta$. If $\mu_p = 1$ for a certain $p \in \Delta$, then $Z_p$ is empty; in other words, the branches of $C$ at $p$ are not split in $\widetilde{C}$.

Let $\widetilde{\omega}$ denote the dualizing sheaf of $\widetilde{C}$. As observed in 2.7, $\widetilde{\omega}$ is the pull–back of $\omega$ to $\widetilde{C}$ and $\omega$ is the push–forward of $\widetilde{\omega}$ to $C$.

For each $p \in \Delta$, each $j = 1, \dots, \mu_p - 1$ and each $m \in \mathbb{Z}$, define the formal sums:

$$\begin{aligned}
Z_p^{(m)} &:= mZ_{p,1} + 2mZ_{p,2} + \cdots + (\mu_p - 1)mZ_{p,\mu_p-1},\\
\widehat{Z}_p^{(m)} &:= (\mu_p - 1)mZ_{p,1} + (\mu_p - 2)mZ_{p,2} + \cdots + mZ_{p,\mu_p-1},\\
Z_p^{(m,j)} &:= Z_p^{(m)} + Z_{p,j+1} + 2Z_{p,j+2} + \cdots + (\mu_p - 1 - j)Z_{p,\mu_p-1},\\
\widehat{Z}_p^{(m,j)} &:= \widehat{Z}_p^{(m)} + Z_{p,j-1} + 2Z_{p,j-2} + \cdots + (j-1)Z_{p,1}.
\end{aligned}$$

Set $Z_p^{(m,\mu_p)} := Z_p^{(m)}$ and $\widehat{Z}_p^{(m,0)} := \widehat{Z}_p^{(m)}$.

**4.3.** *General–position conditions.* We shall eventually assume general–position conditions on effective divisors $D$ of $C$ supported in $\Delta$. Typically, if 4.1 is preserved we shall consider both of the following two conditions:

(4.3.1a) $\quad h^0(\omega_X(-D)) = 0$ if $D$ is effective, $\deg D = g_X$ and $\mathrm{Supp}(D) \subseteq \{x_p |\ p \in \Delta\}$,

(4.3.1b) $\quad h^0(\omega_Y(-D)) = 0$ if $D$ is effective, $\deg D = g_Y$ and $\mathrm{Supp}(D) \subseteq \{y_p |\ p \in \Delta\}$,

whereas if also 4.2 is preserved we shall consider one or both of the following conditions:

(4.3.2a) $$h^0(L_Y(-D))=0 \text{ if } \deg D = g_Y+|I|-|\alpha| \text{ and } 0\leq D\leq \textstyle\sum_{p\in I} y_p,$$

(4.3.2b) $$h^0(M_X(-D))=0 \text{ if } \deg D = g_X+|J|-|\beta| \text{ and } 0\leq D\leq \textstyle\sum_{p\in J} x_p.$$

If Conditions (4.3.1) hold, so do Conditions (4.3.2).

## 5. Degeneration

**Lemma 5.1.** *Preserve* 4.1–2. *Let* $\widetilde{\pi}$ *be a regular smoothing of* $\widetilde{C}$ *and* $\omega_{\widetilde{\pi}}$ *the dualizing sheaf of* $\widetilde{\pi}$. *Put*

$$\mathcal{L}:=\omega_{\widetilde{\pi}}\Big(\sum_{p\in\Delta} Z_p^{(\alpha_p,\rho_p)}+\gamma Y\Big).$$

*If* (4.3.2a) *holds, so do the following five assertions.*

(5.1.1a) $$h^0(\mathcal{L}|_Y)>0,$$

(5.1.1b) $$h^0(\mathcal{L}|_Y(-\textstyle\sum_{p\in I} y_p))=0,$$

(5.1.1c) $$h^0(\mathcal{L}|_X(-\textstyle\sum_{p\in I} x_p))=g-h^0(\mathcal{L}|_Y),$$

(5.1.1d) $$h^0(\mathcal{L}|_{Z_p}(-x_p-y_p))=0 \textit{ for every } p\in\Delta,$$

(5.1.1e) $$h^0(\mathcal{L}|_{Z_p}(-x_p))=h^0(\mathcal{L}|_{Z_p}(-y_p))=0 \textit{ for every } p\in I.$$

*Proof.* Note first that

(5.1.2) $$\mathcal{L}|_X\cong L_X \text{ and } \mathcal{L}|_Y\cong L_Y,$$

where the second isomorphism holds because of (3.1.1a,d). Now, for each $p\in\Delta$ and each $j=1,\dots,\mu_p-1$, let $\omega_{Z_{p,j}}$ denote the dualizing sheaf of $Z_{p,j}$. Then

$$\mathcal{L}|_{Z_{p,j}}\cong\begin{cases}\omega_{Z_{p,j}}((1+\alpha_p)z_{p,j}+(1-\alpha_p)z_{p,j-1}) & \text{if } j<\rho_p,\\ \omega_{Z_{p,j}}((2+\alpha_p)z_{p,j}+(1-\alpha_p)z_{p,j-1}) & \text{if } j=\rho_p,\\ \omega_{Z_{p,j}}((2+\alpha_p)z_{p,j}-\alpha_p z_{p,j-1}) & \text{if } j>\rho_p.\end{cases}$$

Since $Z_{p,j}$ is smooth and rational, $\omega_{Z_{p,j}}\cong\mathcal{O}_{Z_{p,j}}(-2)$. So,

(5.1.3) $$\mathcal{L}|_{Z_{p,j}}\cong\begin{cases}\mathcal{O}_{Z_{p,j}} & \text{if } p\in I \text{ or } j\neq\rho_p,\\ \mathcal{O}_{Z_{p,j}}(1) & \text{if } p\notin I \text{ and } j=\rho_p.\end{cases}$$

Equations (5.1.1d–e) follow immediately.

Since $\alpha\geq 0$ by Lemma 3.1, it follows from (5.1.2) that

(5.1.4) $$h^0(\mathcal{L}|_X)=g_X+|\alpha|+\delta-1.$$

In addition, since $|\alpha|\geq g_Y$ by (3.1.1c), Equation (5.1.1b) follows from (4.3.2a).

By Lemma 3.1, if there is $p\in I$ such that $\alpha_p=0$, then $\alpha=0$, and hence $g_Y=0$. Conversely, if $g_Y=0$ then $\alpha=0$. So, since $|\alpha|<g_Y+|I|$ by (3.1.1c), by (5.1.2) and (4.3.2a),

(5.1.5) $$h^0(\mathcal{L}|_Y)=\begin{cases}g_Y+|I|-|\alpha| & \text{if } g_Y>0,\\ \delta-1 & \text{if } g_Y=0.\end{cases}$$

In any case, we get (5.1.1a).

Since $\alpha \geq 0$, with equality only if $g_Y = 0$, by (5.1.2) and Riemann–Roch,

$$h^0(\mathcal{L}|_X(-\textstyle\sum_{p\in I} x_p)) = \begin{cases} g_X + |\alpha| - |I| + \delta - 1 & \text{if } g_Y > 0, \\ g_X & \text{if } g_Y = 0. \end{cases} \tag{5.1.6}$$

Putting together (5.1.5) and (5.1.6) we get (5.1.1c). □

**Theorem 5.2.** *Preserve* 4.1–2. *Let $\widetilde{\pi}$ be a regular smoothing of $\widetilde{C}$ and $\pi$ the induced smoothing of $C$. Let $\omega_{\widetilde{\pi}}$ be the dualizing sheaf of $\widetilde{\pi}$. Set*

$$\mathcal{L} := \omega_{\widetilde{\pi}}(\sum_{p\in\Delta} Z_p^{(\alpha_p,\rho_p)} + \gamma Y), \tag{5.2.1}$$

$$\mathcal{M} := \omega_{\widetilde{\pi}}(\sum_{p\in\Delta} \widehat{Z}_p^{(\beta_p,\sigma_p)} + \epsilon X). \tag{5.2.2}$$

*Let $V_X$ and $V_Y$ denote the images of the restriction maps,*

$$\tau_X\colon H^0(\mathcal{L}|_{\widetilde{C}}) \to H^0(\mathcal{L}|_X) \quad \textit{and} \quad \tau_Y\colon H^0(\mathcal{M}|_{\widetilde{C}}) \to H^0(\mathcal{M}|_Y).$$

*Then the following three statements hold.*

1. *If* (4.3.2a) *holds, then $\mathcal{L}$ is the canonical sheaf of $\widetilde{\pi}$ with focus on $X$, the map $\tau_X$ is injective, $(V_X, \mathcal{L}|_X)$ is the limit canonical aspect of $\widetilde{\pi}$ with focus on $X$, the point $x_p$ is not a base point of the aspect for any $p \in \Delta$,*

$$\operatorname{codim}(V_X, H^0(\mathcal{L}|_X)) = |\alpha| - g_Y \quad \textit{and} \quad V_X \supseteq H^0(\mathcal{L}|_X(-\textstyle\sum_{p\in I} x_p)).$$

2. *If* (4.3.2b) *holds, then $\mathcal{M}$ is the canonical sheaf of $\widetilde{\pi}$ with focus on $Y$, the map $\tau_Y$ is injective, $(V_Y, \mathcal{M}|_Y)$ is the limit canonical aspect of $\widetilde{\pi}$ with focus on $Y$, the point $y_p$ is not a base point of the aspect for any $p \in \Delta$,*

$$\operatorname{codim}(V_Y, H^0(\mathcal{M}|_Y)) = |\beta| - g_X \quad \textit{and} \quad V_Y \supseteq H^0(\mathcal{M}|_Y(-\textstyle\sum_{p\in J} y_p)).$$

3. *If* (4.3.2a–b) *hold, then the limit Weierstrass scheme $W$ of $\pi$ satisfies*

$$[W] = R_X + R_Y + \sum_{p\in\Delta} g(g-1-\alpha_p-\beta_p)p,$$

*where $R_X$ and $R_Y$ are the ramification divisors of $(V_X, \mathcal{L}|_X)$ and $(V_Y, \mathcal{M}|_Y)$.*

*Proof.* Let's prove Statement 1. Since (4.3.2a) holds, Lemma 5.1 applies. Let $L := \mathcal{L}|_{\widetilde{C}}$. Let's check first that $\tau_X$ is injective. In fact, let $s \in H^0(L)$ such that $s|_X = 0$. Then $s|_{Z_p} \in H^0(L|_{Z_p}(-x_p))$ for every $p \in \Delta$. So $s|_{Z_p} = 0$ for every $p \in I$ by (5.1.1e), and hence $s|_Y \in H^0(L|_Y(-\sum_{p\in I} y_p))$. By (5.1.1b), $s|_Y = 0$ as well. So $s|_{Z_p} \in H^0(L|_{Z_p}(-x_p - y_p))$, and thus $s|_{Z_p} = 0$ for every $p \in \Delta$ by (5.1.1d). Hence $s = 0$. So $\tau_X$ is injective.

Let's prove now that $\mathcal{L}$ is the canonical sheaf of $\widetilde{\pi}$ with focus on $X$. We must check that Conditions 1 and 2 of Theorem 2.2 are verified. Condition 1 holds because $\tau_X$ is injective.

Let's check Condition 2. The limit canonical system of $\widetilde{\pi}$ associated to $\mathcal{L}$ has (affine) rank $g$. Hence, to show that Condition 2 holds it is enough to prove that the kernel $N_E$ of the restriction map $H^0(L) \to H^0(L|_E)$ has dimension strictly less than $g$ for every irreducible component $E$ of $\widetilde{C}$ other than $X$. We divide the proof in three steps.

*Step 1*: We show that $\dim N_Y < g$. In fact, if $s \in N_Y$ then $s|_{Z_p} \in H^0(L|_{Z_p}(-y_p))$ for each $p \in \Delta$. So $s|_{Z_p} = 0$ for each $p \in I$ by (5.1.1e), and hence $s|_X \in H^0(L|_X(-\sum_{p\in I} x_p))$. If $s|_X = 0$ then $s|_{Z_p} = 0$ for every $p \in \Delta$ by (5.1.1d), and hence $s = 0$. So,

$$\dim N_Y \leq h^0(L|_X(-\textstyle\sum_{p\in I} x_p)). \tag{5.2.3}$$

Thus $\dim N_Y < g$ by (5.1.1a) and (5.1.1c).

*Step 2*: Let $p \in \Delta$ and $j \in \mathbb{Z}$ with $0 < j < \mu_p$ such that either $p \in I$ or $\rho_p \leq j < \mu_p$. We show that $\dim N_{Z_{p,j}} < g$. In fact, if $s \in N_{Z_{p,j}}$ then $s|_Y \in H^0(L|_Y(-y_p))$ by (5.1.3). By (5.1.1e), if $s|_Y = 0$ then $s|_{Z_q} = 0$ for every $q \in I$, and thus $s|_X \in H^0(L|_X(-\sum_{q\in I} x_q))$. If $s|_X = 0$ as well, then $s = 0$ by (5.1.1d). So,

$$\dim N_{Z_{p,j}} \leq h^0(L|_Y(-y_p)) + h^0(L|_X(-\textstyle\sum_{q\in I} x_q)).$$

Since $\mathcal{L}|_Y \cong L_Y$, it follows from (4.3.2a) that $h^0(L|_Y(-y_p)) = h^0(L|_Y) - 1$. By (5.1.1c), $\dim N_{Z_{p,j}} < g$.

*Step 3*: Let $p \in \Delta - I$ and $j \in \mathbb{Z}$ such that $0 < j < \rho_p$. We show that $\dim N_{Z_{p,j}} < g$. In fact, if $s \in N_{Z_{p,j}}$ then $s|_X \in H^0(L|_X(-x_p))$ by (5.1.3). If $s|_Y = 0$, then $s|_{Z_q} = 0$ for every $q \in I$ by (5.1.1e), and thus $s|_X \in H^0(L|_X(-x_p - \sum_{q\in I} x_q))$. If $s|_X = 0$ as well, then $s = 0$ by (5.1.1d). So,

$$\dim N_{Z_{p,j}} \leq h^0(L|_Y) + h^0(L|_X(-x_p - \textstyle\sum_{q\in I} x_q)). \tag{5.2.4}$$

By Lemma 3.1, $\alpha \geq 0$, with equality only if $I = \Delta$. So $\alpha \neq 0$ because $p \in \Delta - I$. Since $\mathcal{L}|_X \cong L_X$, we have

$$h^0(L|_X(-x_p - \textstyle\sum_{q\in I} x_q)) = h^0(L|_X(-\textstyle\sum_{q\in I} x_q)) - 1. \tag{5.2.5}$$

So $\dim N_{Z_{p,j}} < g$ by (5.1.1c) and (5.2.4–5).

Condition 2 is checked. So $\mathcal{L}$ is the canonical sheaf of $\widetilde{\pi}$ with focus on $X$.

Let's prove now that $(V_X, L|_X)$ is the limit canonical aspect of $\widetilde{\pi}$ with focus on $X$. Since the aspect has rank $g$, it is enough to show that $h^0(L) \leq g$. By definition of $N_Y$, we have $h^0(L) \leq \dim N_Y + h^0(L|_Y)$. Hence,

$$h^0(L) \leq h^0(L|_X(-\textstyle\sum_{p\in I} x_p)) + h^0(L|_Y)$$

by (5.2.3). So $h^0(L) \leq g$ by (5.1.1c). So $(V_X, L|_X)$ is the limit canonical aspect of $\widetilde{\pi}$ with focus on $X$.

Consider the remaining assertions in Statement 1. Let's check first that

$$V_X \supseteq H^0(L|_X(-\textstyle\sum_{p\in I} x_p)). \tag{5.2.6}$$

In fact, if $s \in N_Y$ then $s|_{Z_p} = 0$ for every $p \in I$ by (5.1.1e). Therefore,

$$\tau_X(N_Y) \subseteq H^0(L|_X(-\textstyle\sum_{p\in I} x_p)). \tag{5.2.7}$$

Now, since $h^0(L) = g$, from the definition of $N_Y$ we get $\dim N_Y \geq g - h^0(L|_Y)$, and hence $\dim N_Y \geq h^0(L|_X(-\sum_{p\in I} x_p))$ by (5.1.1c). So equality holds in (5.2.7), and thus (5.2.6) follows.

Let's check now that $x_p$ is not a base point of $(V_X, L|_X)$ for any $p \in \Delta$. In fact, if there were $p \in I$ such that $V_X \subseteq H^0(L|_X(-x_p))$, then the restriction $H^0(L) \to H^0(L|_{Z_p})$ would be zero by (5.1.1e), thus contradicting Condition 2 of Theorem 2.2. In addition, if $V_X \subseteq H^0(L|_X(-x_p))$ for a certain $p \in \Delta - I$, then

$$h^0(L|_X(-\textstyle\sum_{q\in I} x_q)) = h^0(L|_X(-x_p - \textstyle\sum_{q\in I} x_q))$$

by (5.2.6), thus contradicting (5.2.5). So, no point of $\{x_p | \, p \in \Delta\}$ is a base point of $(V_X, L|_X)$.

Finally, $\operatorname{codim}(V_X, H^0(L|_X)) = |\alpha| - g_Y$ because $\dim V_X = g$, and because of (5.1.4). The proof of Statement 1 is complete. Statement 2 follows by analogy.

Let's prove now Statement 3. By Statements 1 and 2, the sheaves $\mathcal{L}$ and $\mathcal{M}$ are the canonical sheaves and $(V_X, \mathcal{L}|_X)$ and $(V_Y, \mathcal{M}|_Y)$ are the limit canonical aspects of $\widetilde{\pi}$ with foci on $X$ and $Y$, respectively. By (5.2.1), for each $p \in \Delta$ the correction number for $\mathcal{L}$ at $x_p$ is $\alpha_p$. By (5.2.2), for each $p \in \Delta$ the correction number for $\mathcal{M}$ at $y_p$ is $\beta_p$. Then Statement 3 follows from Theorem 2.8. $\square$

**Remark 5.3.** Keep the set–up of Theorem 5.2. Let $L_C$ be the push–forward to $C$ of $\mathcal{L}|_{\widetilde{C}}$ and $M_C$ that of $\mathcal{M}|_{\widetilde{C}}$. It follows from (5.1.3) that the sheaf $L_C$ is torsion–free of generic rank 1, and fails to be invertible precisely along $\Delta - I$. Likewise, $M_C$ is torsion–free of generic rank 1, and fails to be invertible precisely along $\Delta - J$.

## 6. Enriched structures

**6.1.** *Deformation theory.* Let $C$ be a nodal curve defined over a field $k$. Since $C$ is one–dimensional, generically smooth, and a local complete intersection, $\operatorname{Ext}^2_C(\Omega^1_C, \mathcal{O}_C) = 0$ by [DM, Lemma 1.3, p. 79]. By [S], there exists a versal formal deformation $\mathcal{V}$ of $C$ over the base scheme

$$M := \operatorname{Spec}(k[[t_1, \dots, t_e]]), \text{ where } e := \dim_k \operatorname{Ext}^1_C(\Omega^1_C, \mathcal{O}_C).$$

Since $C$ is a curve, $H^2(C, \mathcal{O}_C) = 0$. By [S], the formal scheme $\mathcal{V}$ is the formal completion of a scheme $V$ projective and flat over $M$.

Let $p$ be a node of $C$. So $\widehat{\mathcal{O}}_{C,p} \cong k[[u, v]]/(uv)$. The versal formal deformation of $\widehat{\mathcal{O}}_{C,p}$ is a complete, local $k[[t]]$–algebra $\widehat{\mathcal{O}}$ such that $\widehat{\mathcal{O}}/t\widehat{\mathcal{O}} \cong \widehat{\mathcal{O}}_{C,p}$. More precisely, there is an isomorphism of $k[[t]]$–algebras,

$$\widehat{\mathcal{O}} \cong k[[t, u, v]]/(uv - t).$$

Let $p_1, \dots, p_\delta$ denote the nodes of $C$. For $i = 1, \dots, \delta$, let $\widehat{\mathcal{O}}_i$ denote the versal formal deformation of $\widehat{\mathcal{O}}_{C,p_i}$ over $k[[t]]$. Since global deformations induce local ones, there is a map $\phi_i \colon k[[t]] \to k[[t_1, \dots, t_e]]$ such that

$$\widehat{\mathcal{O}}_{V,p_i} \cong \widehat{\mathcal{O}}_i \otimes_{k[[t]]} k[[t_1, \dots, t_e]].$$

Since $C$ is a curve, by [DM, Prop. 1.5, p. 81], the tensor–product map,

$$\phi_1 \otimes \cdots \otimes \phi_\delta \colon k[[t_1, \dots, t_\delta]] \longrightarrow k[[t_1, \dots, t_e]],$$

is formally smooth. Up to changing variables, we may assume that $\phi_1 \otimes \cdots \otimes \phi_\delta$ is the inclusion map. Thus, there is an isomorphism of $k[[t_1, \dots, t_e]]$–algebras,

$$\widehat{\mathcal{O}}_{V,p_i} \cong k[[t_1, \dots, t_e, u_i, v_i]]/(u_i v_i - t_i),$$

for $i = 1, \dots, \delta$.

**Definition 6.2.** Let $\Upsilon$ be a set and $\lambda \in \mathbb{Z}_\Upsilon$. A subset $P$ of $\Upsilon$ is called $\lambda$–*balanced* if $\lambda_E = \lambda_F$ for all $E, F \in P$. If $P$ is $\lambda$–balanced, set $\lambda_P := \lambda_E$ for (any) $E \in P$.

**6.3.** *Certain fractional ideals.* Let $\widetilde{C}$ be a nodal curve defined over a field $k$. Let $\Upsilon$ denote the set of irreducible components of $\widetilde{C}$. For each $P \subseteq \Upsilon$ set

$$\widetilde{C}_P := \bigcup_{E \in P} E.$$

We say that $\widetilde{C}_P$ is a *subcurve of* $\widetilde{C}$. Let $\overline{C}$ denote the disjoint union of the irreducible components of $\widetilde{C}$.

For each pair $(E, F)$ of distinct irreducible components of $\widetilde{C}$ let $\Delta_{E,F}$ denote the reduced Weil divisor with support $E \cap F$. We shall view $\Delta_{E,F}$ as a Cartier divisor on every subcurve of $\widetilde{C}$ or $\overline{C}$ containing either $E$ or $F$ but not both.

For each $\lambda \in \mathbb{Z}_\Upsilon$ and each $P \subseteq \Upsilon$ define

$$D_{\lambda,P} := \sum_{E \in P} \sum_{F \notin P} (\lambda_F - \lambda_E) \Delta_{E,F} \quad \text{and} \quad N_{\lambda,P} := \mathcal{O}_{\widetilde{C}_P}(D_{\lambda,P}).$$

If $\mathbb{P}$ is a partition of $\Upsilon$, let $\overline{C}_\mathbb{P}$ be the disjoint union of the subcurves $\widetilde{C}_P$ for $P \in \mathbb{P}$, and set

$$\overline{N}_{\lambda,\mathbb{P}} := \bigoplus_{P \in \mathbb{P}} N_{\lambda,P}.$$

Let $\mathbb{T}$ be the total partition of $\Upsilon$. Then $\overline{C}_\mathbb{T} = \overline{C}$. For each $\lambda \in \mathbb{Z}_\Upsilon$ set $\overline{N}_\lambda := \overline{N}_{\lambda,\mathbb{T}}$. Then, for each partition $\mathbb{P}$ of $\Upsilon$ in $\lambda$–balanced subsets, there is a natural injection $\overline{N}_{\lambda,\mathbb{P}} \hookrightarrow \overline{N}_\lambda$ satisfying $\overline{N}_{\lambda,\mathbb{P}} \mathcal{O}_{\overline{C}} = \overline{N}_\lambda$.

Note that

$$\overline{N}_{\ell_1\lambda_1 + \ell_2\lambda_2} = \overline{N}_{\lambda_1}^{\ell_1} \overline{N}_{\lambda_2}^{\ell_2}$$

for all $\ell_1, \ell_2 \in \mathbb{Z}$ and all $\lambda_1, \lambda_2 \in \mathbb{Z}_\Upsilon$. So, $\overline{\mathbb{E}} := \{\overline{N}_\lambda \,|\, \lambda \in \mathbb{Z}_\Upsilon\}$ is a subgroup of the group $\overline{\mathbb{U}}$ of invertible sheaves of fractional ideals of $\overline{C}$.

**6.4.** *Enriched structures.* Preserve 6.3. Let $\widetilde{\pi} : \widetilde{S} \to B$ be a regular smoothing of $\widetilde{C}$. Let $t$ be a parameter of $B$ at its special point. For each $\lambda \in \mathbb{Z}_\Upsilon$ define $\mathcal{N}_\lambda := \mathcal{O}_{\widetilde{S}}(\sum_{E \in \Upsilon} \lambda_E E)$. Then, for each $\lambda$–balanced subset $P \subseteq \Upsilon$ define $\rho_{\lambda,P}$ as the composition

$$\rho_{\lambda,P} \colon \mathcal{N}_\lambda \xrightarrow{\cdot t^{\lambda_P}} \mathcal{O}_{\widetilde{S}}(\textstyle\sum_{E \notin P} (\lambda_E - \lambda_P) E) \longrightarrow N_{\lambda,P},$$

where the second map is the natural surjection.

For each $\lambda \in \mathbb{Z}_\Upsilon$ and each partition $\mathbb{P}$ of $\Upsilon$ in $\lambda$–balanced subsets let

$$\rho_{\lambda,\mathbb{P}} := \bigoplus_{P \in \mathbb{P}} \rho_{\lambda,P}|_{\widetilde{C}} \colon \mathcal{N}_\lambda|_{\widetilde{C}} \longrightarrow \overline{N}_{\lambda,\mathbb{P}}.$$

Let $N_{\lambda,\mathbb{P}}$ be the image of $\rho_{\lambda,\mathbb{P}}$. Since each $\rho_{\lambda,P}$ is surjective, $\rho_{\lambda,\mathbb{P}}$ is an isomorphism onto $N_{\lambda,\mathbb{P}}$ and $N_{\lambda,\mathbb{P}} \mathcal{O}_{\overline{C}_\mathbb{P}} = \overline{N}_{\lambda,\mathbb{P}}$. Note that $N_{\lambda,\mathbb{P}}$ does not depend on the choice of $t$.

For each $\lambda \in \mathbb{Z}_\Upsilon$ set $N_\lambda := N_{\lambda,\mathbb{T}}$. Then, for each partition $\mathbb{P}$ of $\Upsilon$ in $\lambda$–balanced subsets, $\rho_{\lambda,\mathbb{T}}$ factors as $\rho_{\lambda,\mathbb{P}}$ followed by the natural injection $\overline{N}_{\lambda,\mathbb{T}} \hookrightarrow \overline{N}_\lambda$. So $N_{\lambda,\mathbb{P}} = N_\lambda$.

Let $\widetilde{\mathbb{U}}$ denote the group of invertible sheaves of fractional ideals of $\widetilde{C}$. Let

$$\mathbb{E}_{\widetilde{\pi}} := \{N_\lambda \,|\, \lambda \in \mathbb{Z}_\Upsilon\} \subseteq \widetilde{\mathbb{U}}.$$

Note that $N_{\ell_1\lambda_1+\ell_2\lambda_2} = N_{\lambda_1}^{\ell_1}N_{\lambda_2}^{\ell_2}$ for all $\ell_1, \ell_2 \in \mathbb{Z}$ and all $\lambda_1, \lambda_2 \in \mathbb{Z}_\Upsilon$. So $\mathbb{E}_{\widetilde{\pi}}$ is the image of the group homomorphism,

$$e_{\widetilde{\pi}} \colon \mathbb{Z}_\Upsilon \longrightarrow \widetilde{\mathbb{U}}, \text{ defined by } \lambda \mapsto N_\lambda.$$

We say that $e_{\widetilde{\pi}}$ is the *enriched structure on* $\widetilde{C}$ *given by* $\widetilde{\pi}$.

**Theorem 6.5.** *Preserve* 6.3. *For each* $m = 1, \dots, d$, *let* $\lambda_m \in \mathbb{Z}_\Upsilon$ *and let* $N_m$ *be an invertible* $\mathcal{O}_{\widetilde{C}}$*–submodule of* $\overline{N}_{\lambda_m}$ *satisfying* $N_m\mathcal{O}_{\overline{C}} = \overline{N}_{\lambda_m}$. *For each* $\ell \in \mathbb{Z}^d$ *set* $\tau^{(\ell)} := \sum_{i=1}^d \ell_i\lambda_i$. *If*

$$(N_1^{\ell_1} \cdots N_d^{\ell_d})\mathcal{O}_{E\cup F} = N_{\tau^{(\ell)},\{E,F\}} \tag{6.5.1}$$

*for each* $\ell \in \mathbb{Z}^d$ *and all distinct* $E, F \in \Upsilon$ *such that* $\tau_E^{(\ell)} = \tau_F^{(\ell)}$, *then there exists a regular smoothing* $\widetilde{\pi}$ *of* $\widetilde{C}$ *whose enriched structure* $e_{\widetilde{\pi}}$ *satisfies*

$$N_m = e_{\widetilde{\pi}}(\lambda_m) \text{ for } m = 1, \dots, d.$$

*Proof.* Let $p_1, \dots, p_\delta$ denote the reducible nodes of $\widetilde{C}$. Let $V/M$ be the versal formal deformation of $\widetilde{C}$. Then $M = \operatorname{Spec}(R)$, where

$$R := k[[t_1, \dots, t_\delta, s_1, \dots, s_e]]$$

for a certain integer $e$. In addition, we may assume that there is an isomorphism of $R$–algebras,

$$\widehat{O}_{V,p_h} \cong R[[u_h, v_h]]/(u_hv_h - t_h), \tag{6.5.2}$$

for each $h = 1, \dots, \delta$, where $u_h$ and $v_h$ are the local equations of the branches of $\widetilde{C}$ meeting at $p_h$; see 6.1.

For each $h = 1, \dots, \delta$ choose $a_h \in k^*$. Consider the surjection $\phi \colon R \to k[[t]]$, sending each $t_h$ to $a_ht$ and each $s_i$ to $t$. Let $B := \operatorname{Spec}(k[[t]])$ and set $S := V \times_M B$, where $B$ is viewed as a closed subscheme of $M$ by means of $\phi$. Since all $a_h$ are non–zero, the surface $S$ is regular. So $S/B$ is a regular smoothing of $\widetilde{C}$. In addition, the isomorphism (6.5.2) restricts to an isomorphism of $k[[t]]$–algebras,

$$\widehat{O}_{S,p_h} \cong k[[t, u_h, v_h]]/(u_hv_h - a_ht),$$

for each $h = 1, \dots, \delta$.

Recall the notation in 6.4. We need to show that we may choose $a_1, \dots, a_\delta \in k^*$ such that $N_{\lambda_m} = N_m$ for $m = 1, \dots, d$. Fix $h \in \{1, \dots, \delta\}$ and $m \in \{1, \dots, d\}$. For convenience, put

$$p := p_h, \quad u := u_h, \quad v := v_h.$$

Let $E, F \in \Upsilon$ such that $p \in E \cap F$. Suppose that $u$ is the local equation of $E$ at $p$ and $v$ that of $F$. The completion $\widehat{\mathcal{N}}_{\lambda_m,p}$ of $\mathcal{N}_{\lambda_m}$ at $p$ is generated by $(1/u)^{\lambda_{m,E}}(1/v)^{\lambda_{m,F}}$. In addition,

$$\begin{aligned}
\widehat{\rho}_{\lambda_m,\{E\}}((1/u)^{\lambda_{m,E}}(1/v)^{\lambda_{m,F}}) &= (t/u)^{\lambda_{m,E}}(1/v)^{\lambda_{m,F}} = (1/a_h)^{\lambda_{m,E}}(1/v)^{\gamma},\\
\widehat{\rho}_{\lambda_m,\{F\}}((1/u)^{\lambda_{m,E}}(1/v)^{\lambda_{m,F}}) &= (1/u)^{\lambda_{m,E}}(t/v)^{\lambda_{m,F}} = (1/a_h)^{\lambda_{m,F}}(1/u)^{-\gamma},
\end{aligned}$$

where $\gamma := \lambda_{m,F} - \lambda_{m,E}$. For each $b \in k^*$ let

$$\psi_b \colon (1/v)^\gamma\widehat{\mathcal{O}}_{E,p} \oplus u^\gamma\widehat{\mathcal{O}}_{F,p} \longrightarrow k$$

be the map defined by sending $(f(1/v)^\gamma, gu^\gamma)$ to $f(p) - bg(p)$. Then $\widehat{N}_{\lambda_m,p} = \operatorname{Ker}(\psi_{a_h^\gamma})$. In addition, since $N_m$ is an invertible $\mathcal{O}_{\widetilde{C}}$–submodule of $\overline{N}_{\lambda_m}$ such that $N_m\mathcal{O}_{\overline{C}} = \overline{N}_{\lambda_m}$, there

is $b_m \in k^*$ such that $\widehat{N}_{m,p} = \mathrm{Ker}(\psi_{b_m})$. We claim that we may choose $a_h \in k^*$ such that $a_h^\gamma = b_m$, and hence $\widehat{N}_{\lambda_m,p} = \widehat{N}_{m,p}$. In fact, there's clearly $a_h \in k^*$ such that $a_h^\gamma = b_m$ if $\gamma \neq 0$. On the other hand, if $\gamma = 0$ then $\widehat{N}_{m,p} = \widehat{\mathcal{O}}_{\widetilde{C},p}$ by Condition (6.5.1), and hence $b = 1$. So any choice of $a_h \in k^*$ yields $a_h^\gamma = b_m$.

Now, let $m$ vary in $\{1, \dots, d\}$ and pick $b_m \in k^*$ for each $m$ as in the above paragraph. By Condition (6.5.1), $b_1^{\ell_1} \cdots b_d^{\ell_d} = 1$ for every $\ell \in \mathbb{Z}^d$ such that $\tau_E^{(\ell)} = \tau_F^{(\ell)}$. It follows that there is $a_h \in k^*$ such that $b_m = a_h^{\lambda_{m,F} - \lambda_{m,E}}$ for every $m$. Then $\widehat{N}_{\lambda_m,p} = \widehat{N}_{m,p}$ for each $m = 1, \dots, d$.

Finally, let $h$ vary in $\{1, \dots, \delta\}$ and pick $a_h \in k^*$ for each $h$ as in the above paragraph. Now, for each $m = 1, \dots, d$ the sheaves of fractional ideals $N_m$ and $N_{\lambda_m}$ coincide away from the reducible nodes of $\widetilde{C}$ because

$$N_m \mathcal{O}_{\overline{C}} = \overline{N}_{\lambda_m} = N_{\lambda_m} \mathcal{O}_{\overline{C}}.$$

As we proved that $N_{\lambda_m,p} = N_{m,p}$ for each reducible node $p$ of $\widetilde{C}$ and each $m = 1, \dots, d$, we get $N_{\lambda_m} = N_m$ for $m = 1, \dots, d$. □

**Corollary 6.6.** *Preserve* 6.3. *Let $\lambda \in \mathbb{Z}_\Upsilon$ and let $\mathbb{P}$ be the partition of $\Upsilon$ in maximal $\lambda$–balanced subsets. Let $N$ be an invertible sheaf of $\widetilde{C}$. Then there is a regular smoothing $\widetilde{\pi} \colon \widetilde{S} \to B$ of $\widetilde{C}$ such that*

$$N \cong \mathcal{O}_{\widetilde{S}}(\textstyle\sum_{E \in \Upsilon} \lambda_E E)\big|_{\widetilde{C}}$$

*if and only if $N|_{\widetilde{C}_P} \cong N_{\lambda,P}$ for every $P \in \mathbb{P}$.*

*Proof.* The "only if" assertion is clear; see 6.4. Let's prove now the "if" assertion. Since $N|_{\widetilde{C}_P} \cong N_{\lambda,P}$ for every $P \in \mathbb{P}$, we may assume that $N$ is an invertible $\mathcal{O}_{\widetilde{C}}$–submodule of $\overline{N}_{\lambda,\mathbb{P}}$ satisfying $N\mathcal{O}_{\overline{C}_\mathbb{P}} = \overline{N}_{\lambda,\mathbb{P}}$. To finish the proof we apply Theorem 6.5 with $d = 1$, with $\lambda_1 = \lambda$ and $N_1 = N$. So we check that the conditions for Theorem 6.5 hold. First, since $N\mathcal{O}_{\overline{C}_\mathbb{P}} = \overline{N}_{\lambda,\mathbb{P}}$, we have $N\mathcal{O}_{\overline{C}} = \overline{N}_\lambda$ as well. Second, since $d = 1$, we need only verify Condition (6.5.1) for $\ell_1 = 1$. Now, if $E, F \in \Upsilon$ are such that $\lambda_E = \lambda_F$, then there is $P \in \mathbb{P}$ such that $E, F \in P$. Since $N\mathcal{O}_{\widetilde{C}_P} = N_{\lambda,P}$, we have $N\mathcal{O}_{E \cup F} = N_{\lambda,\{E,F\}}$ as well. So Condition (6.5.1) is verified. By Theorem 6.5, there is a regular smoothing $\widetilde{\pi} \colon \widetilde{S} \to B$ of $\widetilde{C}$ such that $N \cong \mathcal{O}_{\widetilde{S}}(\sum_{E \in \Upsilon} \lambda_E E)|_{\widetilde{C}}$. □

**Remark 6.7.** Preserve 6.3 and 6.4. Mainò calls an enriched structure the canonical generators of the image of $e_{\widetilde{\pi}}$, and view them abstractly, without the additional structure of sheaves of fractional ideals. In [M, Prop. 3.16] she gives an intrinsic characterization of the enriched structures a nodal curve can have. The proof of Theorem 6.5 was inspired by the proof given to [M, Prop. 3.16]. In [M] Mainò constructs and studies the moduli of stable curves with enriched structures.

## 7. Regeneration, I

**Theorem 7.1.** *Preserve* 4.1–2. *Let $L$ and $M$ be invertible sheaves on $\widetilde{C}$. Then the following three statements hold.*

1. *If* (4.3.2a) *holds and* $g_Y > 0$, *then* $L$ *is the restriction of the canonical sheaf with focus on* $X$ *of some regular smoothing of* $\widetilde{C}$ *if and only if*

$$L|_X \cong L_X, \quad L|_Y \cong L_Y, \quad L|_{Z_{p,j}} \cong \begin{cases} \mathcal{O}_{Z_{p,j}} & \text{if } p \in I \text{ or } j \neq \rho_p, \\ \mathcal{O}_{Z_{p,j}}(1) & \text{if } p \notin I \text{ and } j = \rho_p. \end{cases} \tag{7.1.1}$$

2. *If* (4.3.2b) *holds and* $g_X > 0$, *then* $M$ *is the restriction of the canonical sheaf with focus on* $Y$ *of some regular smoothing of* $\widetilde{C}$ *if and only if*

$$M|_X \cong M_X, \quad M|_Y \cong M_Y, \quad M|_{Z_{p,j}} \cong \begin{cases} \mathcal{O}_{Z_{p,j}} & \text{if } p \in J \text{ or } j \neq \sigma_p, \\ \mathcal{O}_{Z_{p,j}}(1) & \text{if } p \notin J \text{ and } j = \sigma_p. \end{cases} \tag{7.1.2}$$

3. *If* (4.3.2a–b) *hold and* $g_X g_Y > 0$, *then* $L$ *and* $M$ *are the restrictions of the canonical sheaves with foci on* $X$ *and* $Y$, *respectively, of some regular smoothing of* $\widetilde{C}$ *if and only if* (7.1.1–2) *hold and* $(L^{\otimes\widetilde{\beta}} \otimes M^{\otimes\widetilde{\alpha}})|_{\widetilde{C}_{I\cap J}} \cong \widetilde{K}_{I\cap J}$, *where*

$$\widetilde{C}_{I\cap J} := X \cup Y \cup (\bigcup_{p\in I\cap J} Z_p) \subseteq \widetilde{C}$$

*and*

$$\widetilde{K}_{I\cap J} := \widetilde{\omega}|^{\otimes(\widetilde{\alpha}+\widetilde{\beta})}_{\widetilde{C}_{I\cap J}}(\sum_{p\notin I\cap J}(\widetilde{\beta}\alpha_p - \widetilde{\alpha}\beta_p)(x_p - y_p) - \widetilde{\alpha}\sum_{p\notin J} x_p - \widetilde{\beta}\sum_{p\notin I} y_p). \tag{7.1.3}$$

*Proof.* Let $\Upsilon$ be the set of irreducible components of $\widetilde{C}$. Let $\lambda, \nu \in \mathbb{Z}_\Upsilon$ such that

$$\sum_{p\in\Delta} Z_p^{(\alpha_p,\rho_p)} + \gamma Y = \sum_{E\in\Upsilon} \lambda_E E \quad \text{and} \quad \sum_{p\in\Delta} \widehat{Z}_p^{(\beta_p,\sigma_p)} + \epsilon X = \sum_{E\in\Upsilon} \nu_E E. \tag{7.1.4}$$

Let's prove Statement 1. Consider first the "only if" assertion. Let $\widetilde{\pi}$ be a regular smoothing of $\widetilde{C}$ and $\mathcal{L} := \omega_{\widetilde{\pi}}(\sum_{p\in\Delta} Z_p^{(\alpha_p,\rho_p)} + \gamma Y)$, where $\omega_{\widetilde{\pi}}$ is the dualizing sheaf of $\widetilde{\pi}$. By Theorem 5.2, $\mathcal{L}$ is the canonical sheaf with focus on $X$. If $L \cong \mathcal{L}|_{\widetilde{C}}$, then (7.1.1) holds by (5.1.2–3).

Consider now the "if" assertion. Preserve 6.3. Let $\mathbb{P}$ denote the partition of $\Upsilon$ in maximal $\lambda$–balanced subsets. Since $g_Y > 0$, by Lemma 3.1, $\alpha_p > 0$ for every $p \in I$. Hence $\gamma > 0$ as well. So $\lambda_X \neq \lambda_Y$, and hence $\widetilde{C}_P$ is treelike for every $P \in \mathbb{P}$. Let $N := L \otimes \widetilde{\omega}^{-1}$. Since (7.1.1) holds and $\widetilde{C}_P$ is treelike, $N|_{\widetilde{C}_P} \cong N_{\lambda,P}$ for every $P \in \mathbb{P}$. By Corollary 6.6, there is a regular smoothing $\widetilde{\pi}\colon \widetilde{S} \to B$ of $\widetilde{C}$ such that $N \cong \mathcal{N}|_{\widetilde{C}}$, where $\mathcal{N} := \mathcal{O}_{\widetilde{S}}(\sum_{E\in\Upsilon} \lambda_E E)$. Let $\omega_{\widetilde{\pi}}$ be the dualizing sheaf of $\widetilde{\pi}$. Since (7.1.4) holds, by Theorem 5.2 the canonical sheaf $\mathcal{L}$ of $\widetilde{\pi}$ with focus on $X$ satisfies $\mathcal{L} \cong \omega_{\widetilde{\pi}} \otimes \mathcal{N}$. So $L \cong \mathcal{L}|_{\widetilde{C}}$. The proof of Statement 1 is complete. Statement 2 follows by analogy.

Let's prove Statement 3. For each pair $(\ell, m) \in \mathbb{Z} \times \mathbb{Z}$ let $\tau^{(\ell,m)} := \ell\lambda + m\nu$. Then $\tau_X^{(\ell,m)} = \tau_Y^{(\ell,m)}$ if and only if $(\ell, m) \in \mathbb{Z}(\widetilde{\beta}, \widetilde{\alpha})$. In addition, using that

$$\gamma = \mu_p(\alpha_p + 1) - \rho_p \text{ and } \epsilon = \mu_p\beta_p + \sigma_p \text{ for every } p \in \Delta,$$

it follows from (7.1.4) that

$$\sum_{E\in\Upsilon} \tau_E^{(\widetilde{\beta},\widetilde{\alpha})} E = \frac{\gamma\epsilon}{\gcd(\gamma,\epsilon)}(X + Y + \sum_{p\in I\cap J}\sum_{j=1}^{\mu_p-1} Z_{p,j}) + D_1 + D_2 + D_3, \tag{7.1.5}$$

where

$$D_1 = \sum_{p\in J\setminus I} (\widetilde{\alpha}\epsilon + \widetilde{\beta}\alpha_p - \widetilde{\alpha}\beta_p) Z_{p,1} + \sum_{p\in I\setminus J} (\widetilde{\beta}\gamma + \widetilde{\alpha}\beta_p - \widetilde{\beta}\alpha_p) Z_{p,\mu_p-1},$$
$$D_2 = \sum_{p\notin J} (\widetilde{\alpha}\epsilon + \widetilde{\beta}\alpha_p - \widetilde{\alpha}(\beta_p+1)) Z_{p,1} + \sum_{p\notin I} (\widetilde{\beta}\gamma + \widetilde{\alpha}\beta_p - \widetilde{\beta}(\alpha_p+1)) Z_{p,\mu_p-1},$$

and where $D_3$ is a formal sum with integer coefficients of the rational curves $Z_{p,j}$ with $p \notin I \cap J$ and $1 < j < \mu_p - 1$.

Consider first the “only if” assertion of Statement 3. Let $\widetilde{\pi}$ be a regular smoothing of $\widetilde{C}$, and $\omega_{\widetilde{\pi}}$ its dualizing sheaf. By Theorem 5.2, the canonical sheaves $\mathcal{L}$ and $\mathcal{M}$ of $\widetilde{\pi}$ with foci on $X$ and $Y$, respectively, are given by (5.2.1–2). By (7.1.4),

$$\mathcal{L}^{\otimes\widetilde{\beta}} \otimes \mathcal{M}^{\otimes\widetilde{\alpha}} \cong \omega_{\widetilde{\pi}}^{\otimes(\widetilde{\alpha}+\widetilde{\beta})}(\sum_{E\in\Upsilon} \tau_E^{(\widetilde{\beta},\widetilde{\alpha})} E).$$

So, it follows from (7.1.5) that $\mathcal{L}^{\otimes\widetilde{\beta}} \otimes \mathcal{M}^{\otimes\widetilde{\alpha}}|_{\widetilde{C}_{I\cap J}} \cong \widetilde{K}_{I\cap J}$.

Consider now the “if” assertion. Let $\widetilde{L} := L \otimes \widetilde{\omega}^{-1}$ and $\widetilde{M} := M \otimes \widetilde{\omega}^{-1}$. We shall use the set–up of 6.3. Let $\overline{L} := \overline{N}_\lambda$ and $\overline{M} := \overline{N}_\nu$. By (7.1.1–2), we may view $\widetilde{L}$ and $\widetilde{M}$ as invertible $\mathcal{O}_{\widetilde{C}}$–submodules of $\overline{L}$ and $\overline{M}$ such that $\widetilde{L}\mathcal{O}_{\overline{C}} = \overline{L}$ and $\widetilde{M}\mathcal{O}_{\overline{C}} = \overline{M}$. Multiplication by $c \in k^*_\Upsilon$ gives automorphisms of $\overline{L}$ and $\overline{M}$; let $\widetilde{L}_c$ and $\widetilde{M}_c$ denote the images of $\widetilde{L}$ and $\widetilde{M}$ under these automorphisms. It's clear that $\widetilde{L}_c\mathcal{O}_{\overline{C}} = \overline{L}$ and $\widetilde{M}_c\mathcal{O}_{\overline{C}} = \overline{M}$. For each $c \in k^*_\Upsilon$, each $p \in \Delta$ and each $j = 0, \dots, \mu_p$ let $c_{p,j} := c_{Z_{p,j}}$.

We'll choose $c, d \in k^*_\Upsilon$ such that $\widetilde{L}_c$ and $\widetilde{M}_d$ satisfy Condition (6.5.1) of Theorem 6.5. In other words, we'll pick $c, d \in k^*_\Upsilon$ such that

$$\widetilde{L}_c^\ell \widetilde{M}_d^m \mathcal{O}_{E\cup F} = N_{\tau^{(\ell,m)},\{E,F\}} \tag{7.1.6}$$

for all $(\ell, m) \in \mathbb{Z}\times\mathbb{Z}$ and all distinct $E, F \in \Upsilon$ such that $\tau_E^{(\ell,m)} = \tau_F^{(\ell,m)}$.

In fact, we need only obtain (7.1.6) for $E$ and $F$ that intersect, that is, for $E$ and $F$ such that $\{E, F\} = \{Z_{p,j-1}, Z_{p,j}\}$ for some $p \in \Delta$ and some $j = 1, \dots, \mu_p$. In addition, we need only obtain (7.1.6) for $(\ell, m)$ in a $\mathbb{Z}$–basis of

$$H_{E,F} := \{(\ell, m) \in \mathbb{Z}\times\mathbb{Z} \,|\, \tau_E^{(\ell,m)} = \tau_F^{(\ell,m)}\}.$$

Let $H_{p,j} := H_{Z_{p,j-1},Z_{p,j}}$ for each $p \in \Delta$ and each $j = 1, \dots, \mu_p$. Choose a $\mathbb{Z}$–basis $B_{p,j}$ of $H_{p,j}$ for each $p \in \Delta$ and each $j = 1, \dots, \mu_p$. Note that $H_{p,j} = \mathbb{Z}\times\mathbb{Z}$ if and only if $\alpha_p = \beta_p = 0$ and $\sigma_p < j \le \rho_p$; in this case let $B_{p,j} := \{(1, 0), (0, 1)\}$. In all other cases, $H_{p,j}$ has rank 1. If $p \in I \cap J$ then $(\widetilde{\beta}, \widetilde{\alpha})$ is a basis of $H_{p,j}$ for every $j$; in this case let $B_{p,j} := \{(\widetilde{\beta}, \widetilde{\alpha})\}$.

Consider distinct $E, F \in \Upsilon$ with non–empty intersection. Then either $E \cup F$ is treelike or $\{E, F\} = \{X, Y\}$ and there are distinct $p, q \in \Delta$ such that $\mu_p = \mu_q = 1$. In either case there is an isomorphism between $\widetilde{L}^\ell \widetilde{M}^m \mathcal{O}_{E\cup F}$ and $N_{\tau^{(\ell,m)},\{E,F\}}$ for each $(\ell, m) \in H_{E,F}$. In fact, such an isomorphism exists in the former case because both sheaves have the same restriction to $E$ and to $F$; and in the latter case because $H_{X,Y} = \mathbb{Z}(\widetilde{\beta}, \widetilde{\alpha})$ and $L^{\otimes\widetilde{\beta}} \otimes M^{\otimes\widetilde{\alpha}}|_{X\cup Y} \cong \widetilde{K}_{I\cap J}|_{X\cup Y}$. So, for each $(\ell, m) \in H_{E,F}$ there is a unique $e_{E,F}^{(\ell,m)} \in k^*$ such that

$$\widetilde{L}^\ell \widetilde{M}^m \mathcal{O}_{E\cup F} = (1, e_{E,F}^{(\ell,m)}) N_{\tau^{(\ell,m)},\{E,F\}}$$

as subsheaves of $N_{\tau^{(\ell,m)},E} \oplus N_{\tau^{(\ell,m)},F}$. Let $e^{(\ell,m)}_{p,j} := e^{(\ell,m)}_{Z_{p,j-1},Z_{p,j}}$ for each $p \in \Delta$ and each $j = 1, \ldots, \mu_p$. Since $L^{\otimes\widetilde{\beta}} \otimes M^{\otimes\widetilde{\alpha}}|_{\widetilde{C}_{I\cap J}} \cong \widetilde{K}_{I\cap J}$, it follows that

$$e^{(\widetilde{\beta},\widetilde{\alpha})}_{p,1} e^{(\widetilde{\beta},\widetilde{\alpha})}_{p,2} \cdots e^{(\widetilde{\beta},\widetilde{\alpha})}_{p,\mu_p} = e^{(\widetilde{\beta},\widetilde{\alpha})}_{q,1} e^{(\widetilde{\beta},\widetilde{\alpha})}_{q,2} \cdots e^{(\widetilde{\beta},\widetilde{\alpha})}_{q,\mu_q} \text{ for all } p, q \in I \cap J. \tag{7.1.7}$$

Therefore, finding $c, d \in k^*_\Upsilon$ such that (7.1.6) holds for all $(\ell, m) \in \mathbb{Z} \times \mathbb{Z}$ and all distinct $E, F \in \Upsilon$ with $\tau^{(\ell,m)}_E = \tau^{(\ell,m)}_F$ is equivalent to finding $c, d \in k^*_\Upsilon$ such that, for each $p \in \Delta$,

$$c^\ell_{p,j-1} d^m_{p,j-1} = e^{(\ell,m)}_{p,j} c^\ell_{p,j} d^m_{p,j} \text{ for each } j = 1, \ldots, \mu_p \text{ and each } (\ell, m) \in B_{p,j}. \tag{7.1.8}$$

Set $c_X := 1$ and $d_X := 1$. For $p \in I\cap J$ choose $c_{p,j}, d_{p,j} \in k^*$ inductively for $j = 1, \ldots, \mu_p-1$ such that

$$e^{(\widetilde{\beta},\widetilde{\alpha})}_{p,j} c^{\widetilde{\beta}}_{p,j} d^{\widetilde{\alpha}}_{p,j} = c^{\widetilde{\beta}}_{p,j-1} d^{\widetilde{\alpha}}_{p,j-1}.$$

By (7.1.7), we may choose $c_Y, d_Y \in k^*$ such that

$$e^{(\widetilde{\beta},\widetilde{\alpha})}_{p,\mu_p} c^{\widetilde{\beta}}_Y d^{\widetilde{\alpha}}_Y = c^{\widetilde{\beta}}_{p,\mu_p-1} d^{\widetilde{\alpha}}_{p,\mu_p-1}$$

for every $p \in I \cap J$. If $I \cap J = \emptyset$, set $c_Y := 1$ and $d_Y := 1$. Then (7.1.8) is achieved for every $p \in I \cap J$.

If $p \in \Delta - (I \cap J)$ is such that $\alpha_p = \beta_p = 0$ and $\sigma_p < \rho_p$, then choose $c_{p,j}$ inductively for $j = 1, \ldots, \rho_p$ such that

$$e^{(1,0)}_{p,j} c_{p,j} = c_{p,j-1},$$

and choose $d_{p,j}$ inductively for $j = \mu_p - 1, \ldots, \sigma_p$ such that

$$d_{p,j} = e^{(0,1)}_{p,j+1} d_{p,j+1}.$$

In addition, set $c_{p,j} := 1$ for $j = \rho_p + 1, \ldots, \mu_p - 1$ and $d_{p,j} := 1$ for $j = 1, \ldots, \sigma_p - 1$. Then (7.1.8) is achieved.

Let $p \in \Delta - (I \cap J)$ such that either $\alpha_p \neq 0$ or $\beta_p \neq 0$ or $\sigma_p \geq \rho_p$. Then $H_{p,j}$ has rank 1 for every $j = 1, \ldots, \mu_p$. By (7.1.5), $\tau^{(\widetilde{\beta},\widetilde{\alpha})}_X = \tau^{(\widetilde{\beta},\widetilde{\alpha})}_Y$. Now, since $\widetilde{\alpha}$ and $\widetilde{\beta}$ are positive and $p \notin I \cap J$, it follows from (7.1.5) as well that either $\tau^{(\widetilde{\beta},\widetilde{\alpha})}_{Z_{p,1}} \neq \tau^{(\widetilde{\beta},\widetilde{\alpha})}_X$ or $\tau^{(\widetilde{\beta},\widetilde{\alpha})}_{Z_{p,\mu_p-1}} \neq \tau^{(\widetilde{\beta},\widetilde{\alpha})}_Y$. So, there is $h_p \in \{1, \ldots, \mu_p - 1\}$ such that

$$H_{p,h_p} \cap H_{p,h_p+1} = 0. \tag{7.1.9}$$

For $j = 1, \ldots, h_p - 1$ choose $c_{p,j}, d_{p,j} \in k^*$ inductively such that

$$e^{(\ell,m)}_{p,j} c^\ell_{p,j} d^m_{p,j} = c^\ell_{p,j-1} d^m_{p,j-1} \text{ where } (\ell, m) \in B_{p,j}.$$

For $j = \mu_p - 1, \ldots, h_p + 1$ choose $c_{p,j}, d_{p,j} \in k^*$ inductively such that

$$c^\ell_{p,j} d^m_{p,j} = e^{(\ell,m)}_{p,j+1} c^\ell_{p,j+1} d^m_{p,j+1} \text{ where } (\ell, m) \in B_{p,j+1}.$$

Now, let $(\ell_-, m_-) \in B_{p,h_p}$ and $(\ell_+, m_+) \in B_{p,h_p+1}$. By (7.1.9), $(\ell_-, m_-)$ and $(\ell_+, m_+)$ are $\mathbb{Z}$–independent. So, we may choose $c_{p,h_p}, d_{p,h_p} \in k^*$ such that

$$e^{(\ell_-,m_-)}_{p,h_p} c^{\ell_-}_{p,h_p} d^{m_-}_{p,h_p} = c^{\ell_-}_{p,h_p-1} d^{m_-}_{p,h_p-1} \quad \text{and} \quad c^{\ell_+}_{p,h_p} d^{m_+}_{p,h_p} = e^{(\ell_+,m_+)}_{p,h_p+1} c^{\ell_+}_{p,h_p+1} d^{m_+}_{p,h_p+1}.$$

Then (7.1.8) is achieved.

So, we obtained $c, d \in k_\Upsilon^*$ such that (7.1.6) holds for all $(\ell, m) \in \mathbb{Z} \times \mathbb{Z}$ and all distinct $E, F \in \Upsilon$ such that $\tau_E^{(\ell,m)} = \tau_F^{(\ell,m)}$. By Theorem 6.5 there is a regular smoothing $\widetilde{\pi}\colon \widetilde{S} \to B$ of $\widetilde{C}$ such that

$$\widetilde{L}_c \cong \mathcal{O}_{\widetilde{S}}(\sum_{E\in\Upsilon} \lambda_E E)|_{\widetilde{C}} \quad \text{and} \quad \widetilde{M}_d \cong \mathcal{O}_{\widetilde{S}}(\sum_{E\in\Upsilon} \nu_E E)|_{\widetilde{C}}.$$

Since $\widetilde{L}_c \cong L \otimes \widetilde{\omega}^{-1}$ and $\widetilde{M}_d \cong M \otimes \widetilde{\omega}^{-1}$, it follows from (7.1.4) and Theorem 5.2 that $L$ and $M$ are the restrictions of the canonical sheaves of $\widetilde{\pi}$ with foci on $X$ and $Y$, respectively. ☐

## 8. Regeneration, II

**Definition 8.1.** Preserve 4.1–2. Let $V \subseteq H^0(L_X)$ and $W \subseteq H^0(M_Y)$ be vector subspaces.

Suppose (4.3.2a) holds. Then $V$ is called *$\mu$–smoothable* if $(V, L_X)$ is the limit canonical aspect with focus on $X$ of a regular smoothing of $\widetilde{C}$. Let $\mathbb{V}_{\mu,X} \subseteq \mathbb{G}_X$ be the subset of $\mu$–smoothable subspaces.

Suppose (4.3.2b) holds. Then $W$ is called *$\mu$–smoothable* if $(W, M_Y)$ is the limit canonical aspect with focus on $Y$ of a regular smoothing of $\widetilde{C}$. Let $\mathbb{V}_{\mu,Y} \subseteq \mathbb{G}_Y$ be the subset of $\mu$–smoothable subspaces.

Suppose (4.3.2a–b) hold. Then the pair $(V, W)$ is called *$\mu$–smoothable* if $(V, L_X)$ and $(W, M_Y)$ are the limit canonical aspects with foci on $X$ and $Y$ of the same regular smoothing of $\widetilde{C}$. Let $\mathbb{V}_\mu \subseteq \mathbb{V}_{\mu,X} \times \mathbb{V}_{\mu,Y}$ be the subset of $\mu$–smoothable pairs.

**Theorem 8.2.** *Preserve* 4.1–2. *For each $F \subseteq \Delta$ let $C_F$ be the blow–up of $C$ along $\Delta - F$. Let $V \subseteq H^0(L_X)$ and $W \subseteq H^0(M_Y)$ be vector subspaces. Then the following three statements hold.*

1. *If* (4.3.2a) *holds and $g_Y > 0$, then $V \in \mathbb{V}_{\mu,X}$ if and only if there is an invertible sheaf $L$ on $C_I$ such that $L|_X \cong L_X$ and $L|_Y \cong L_Y$ and such that $V$ is the image of the restriction map $H^0(L) \to H^0(L_X)$.*
2. *If* (4.3.2b) *holds and $g_X > 0$, then $W \in \mathbb{V}_{\mu,Y}$ if and only if there is an invertible sheaf $M$ on $C_J$ such that $M|_X \cong M_X$ and $M|_Y \cong M_Y$ and such that $W$ is the image of the restriction map $H^0(M) \to H^0(M_Y)$.*
3. *If* (4.3.2a–b) *hold and $g_X g_Y > 0$, then $(V, W) \in \mathbb{V}_\mu$ if and only if there are an invertible sheaf $L$ on $C_I$ as in Statement 1 and an invertible sheaf $M$ on $C_J$ as in Statement 2 such that their pull–backs $L_{I\cap J}$ and $M_{I\cap J}$ to $C_{I\cap J}$ satisfy*

$$L_{I\cap J}^{\otimes\widetilde{\beta}} \otimes M_{I\cap J}^{\otimes\widetilde{\alpha}} \cong K_{I\cap J}, \tag{8.2.1}$$

   *where*

$$K_{I\cap J} := \omega_{I\cap J}^{\otimes(\widetilde{\alpha}+\widetilde{\beta})}(\sum_{p\notin I\cap J} (\widetilde{\beta}\alpha_p - \widetilde{\alpha}\beta_p)(x_p - y_p) - \widetilde{\alpha}\sum_{p\notin J} x_p - \widetilde{\beta}\sum_{p\notin I} y_p), \tag{8.2.2}$$

   *where $\omega_{I\cap J}$ is the pull–back to $C_{I\cap J}$ of the dualizing sheaf $\omega$ of $C$.*

*Proof.* Let $\psi\colon \widetilde{C} \to C$ be the map contracting the subcurve $Z_p$ for every $p \in \Delta$. For each $F \subseteq \Delta$ let $\varphi_F\colon C_F \to C$ be the blow–up map,

$$\widetilde{C}_F := X \cup Y \cup (\textstyle\bigcup_{p\in F} Z_p) \subseteq \widetilde{C},$$

and let $\psi_F\colon \widetilde{C}_F \to C_F$ be the map contracting the subcurve $Z_p$ for every $p \in F$.

Let's prove Statement 1. First, let $\widetilde{\pi}$ be a regular smoothing of $\widetilde{C}$ such that $(V, L_X)$ is the limit canonical aspect of $\widetilde{\pi}$ with focus on $X$. Let $\widetilde{L}$ be the restriction to $\widetilde{C}$ of the canonical sheaf of $\widetilde{\pi}$ with focus on $X$. By Proposition 7.1,

$$\widetilde{L}|_X \cong L_X, \quad \widetilde{L}|_Y \cong L_Y, \tag{8.2.3a}$$

$$\widetilde{L}|_{Z_{p,j}} \cong \begin{cases} \mathcal{O}_{Z_{p,j}} & \text{if } p \in I \text{ or } j \neq \rho_p, \\ \mathcal{O}_{Z_{p,j}}(1) & \text{if } p \notin I \text{ and } j = \rho_p. \end{cases} \tag{8.2.3b}$$

By (8.2.3b), the sheaf $\psi_*\widetilde{L}$ is torsion–free, rank–1, and fails to be invertible precisely along the set $\Delta - I$. Hence there are an invertible sheaf $L$ on $C_I$ and an isomorphism $\lambda\colon \psi_*\widetilde{L} \to \varphi_{I*}L$. Restricting $\lambda$ to $X$ and $Y$, and removing torsion, we obtain isomorphisms $\widetilde{L}|_X \to L|_X$ and $\widetilde{L}|_Y \to L|_Y$. So there is a commutative diagram,

$$\begin{array}{ccc} H^0(\widetilde{L}) & \longrightarrow & H^0(L) \\ \big\downarrow & & \big\downarrow \\ H^0(\widetilde{L}|_X) & \longrightarrow & H^0(L|_X), \end{array} \tag{8.2.4}$$

where the vertical maps are restriction maps, and the horizontal maps are the isomorphisms induced by $\lambda$. In addition, $L|_X \cong L_X$ and $L|_Y \cong L_Y$ by (8.2.3a). Now, since (8.2.4) is commutative, the restriction maps $\tau\colon H^0(L) \to H^0(L_X)$ and $\widetilde{\tau}\colon H^0(\widetilde{L}) \to H^0(L_X)$ have the same image. Hence $V = \mathrm{Im}(\widetilde{\tau})$ by Theorem 5.2, because $(V, L_X)$ is the limit canonical aspect of $\widetilde{\pi}$ with focus on $X$. So $V = \mathrm{Im}(\tau)$.

Conversely, let $L$ be an invertible sheaf on $C_I$ as in Statement 1. Then $\varphi_{I*}L$ is torsion–free, rank–1, and fails to be invertible precisely along $\Delta - I$. As $\mu_p > 1$ for each $p \in \Delta - I$, there are an invertible sheaf $\widetilde{L}$ on $\widetilde{C}$ such that (8.2.3b) holds and an isomorphism $\lambda\colon \psi_*\widetilde{L} \to \varphi_{I*}L$. As before, $\lambda$ induces isomorphisms $\widetilde{L}|_X \cong L|_X$ and $\widetilde{L}|_Y \cong L|_Y$. So (8.2.3a) holds. By Theorem 7.1, there is a regular smoothing of $\widetilde{C}$ whose canonical sheaf with focus on $X$ restricts to $\widetilde{L}$ on $\widetilde{C}$. As before, $V$ is the image of the restriction map $H^0(\widetilde{L}) \to H^0(L_X)$. So $V$ is $\mu$–smoothable. The proof of Statement 1 is complete. Statement 2 follows by analogy.

Let's prove Statement 3. First, let $\widetilde{\pi}$ be a regular smoothing of $\widetilde{C}$ such that $(V, L_X)$ and $(W, M_Y)$ are the limit canonical aspects of $\widetilde{\pi}$ with foci on $X$ and $Y$, respectively. Let $\widetilde{L}$ and $\widetilde{M}$ be the restrictions to $\widetilde{C}$ of the canonical sheaves of $\widetilde{\pi}$ with foci on $X$ and $Y$, respectively. As observed in the proof of Statement 1, there are invertible sheaves $L$ and $M$ on $C_I$ and $C_J$, respectively, such that $\psi_*\widetilde{L} \cong \varphi_{I*}L$ and $\psi_*\widetilde{M} \cong \varphi_{J*}M$. In addition, $L$ and $M$ satisfy Statements 1 and 2, respectively.

Since $\psi_*\widetilde{L} \cong \varphi_{I*}L$, there is a map $\psi_I^*L \to \widetilde{L}|_{\widetilde{C}_I}$. This map is injective because it is injective generically on $X$ and $Y$ and because $L$ is invertible. So $\psi_I^*L$ and $\widetilde{L}|_{\widetilde{C}_I}$ are isomorphic because they restrict to isomorphic sheaves on the irreducible components of $\widetilde{C}_I$. Analogously, $\psi_J^*M \cong \widetilde{M}|_{\widetilde{C}_J}$. Let $L_{I\cap J}$ and $M_{I\cap J}$ be the pull–backs of $L$ and $M$ to $C_{I\cap J}$. Let $\widetilde{K}_{I\cap J}$ be given by (7.1.3). Since $(\widetilde{L}^{\otimes\widetilde{\beta}} \otimes \widetilde{M}^{\otimes\widetilde{\alpha}})|_{\widetilde{C}_{I\cap J}} \cong \widetilde{K}_{I\cap J}$ by Theorem 7.1, we have

$$\psi_{I\cap J}^*(L_{I\cap J}^{\otimes\widetilde{\beta}} \otimes M_{I\cap J}^{\otimes\widetilde{\alpha}}) \cong \widetilde{K}_{I\cap J}. \tag{8.2.5}$$

Now, since $\widetilde{\omega} \cong \psi^*\omega$, it follows that $\widetilde{K}_{I\cap J} \cong \psi_{I\cap J}^*K_{I\cap J}$. So (8.2.1) holds.

Conversely, let $L$ be an invertible sheaf on $C_I$ and $M$ an invertible sheaf on $C_J$ as in Statement 3. As observed in the proof of Statement 1, there are invertible sheaves $\widetilde{L}$ and $\widetilde{M}$ on $\widetilde{C}$ such that $\psi_*\widetilde{L} \cong \varphi_{I*}L$ and $\psi_*\widetilde{M} \cong \varphi_{J*}M$, and such that (8.2.3) and (8.2.6) below hold.

$$\widetilde{M}|_X \cong M_X, \quad \widetilde{M}|_Y \cong M_Y, \tag{8.2.6a}$$

$$\widetilde{M}|_{Z_{p,j}} \cong \begin{cases} \mathcal{O}_{Z_{p,j}} & \text{if } p \in J \text{ or } j \neq \sigma_p, \\ \mathcal{O}_{Z_{p,j}}(1) & \text{if } p \notin J \text{ and } j = \sigma_p. \end{cases} \tag{8.2.6b}$$

Now, since $\psi^*_{I\cap J}K_{I\cap J} \cong \widetilde{K}_{I\cap J}$, it follows from (8.2.1) that (8.2.5) holds. As shown above, $\psi^*_I L \cong \widetilde{L}|_{\widetilde{C}_I}$ and $\psi^*_J M \cong \widetilde{M}|_{\widetilde{C}_J}$. Hence $(\widetilde{L}^{\otimes\widetilde{\beta}} \otimes \widetilde{M}^{\otimes\widetilde{\alpha}})|_{\widetilde{C}_{I\cap J}} \cong \widetilde{K}_{I\cap J}$. By Theorem 7.1, there is a regular smoothing of $\widetilde{C}$ whose canonical sheaves with foci on $X$ and $Y$ restrict to $\widetilde{L}$ and $\widetilde{M}$, respectively. As shown in the proof of Statement 1, the restriction maps $H^0(\widetilde{L}) \to H^0(L_X)$ and $H^0(\widetilde{M}) \to H^0(M_Y)$ have images $V$ and $W$, respectively. So $(V, W)$ is $\mu$–smoothable. ☐

**8.3.** *Tori actions.* Preserve 4.1–2. Denote by $C_{I\cap J}$ the blow–up of $C$ along the points of $\Delta - (I \cap J)$. Fix isomorphisms $\zeta_{X,p}\colon L_X(x_p) \to k$ and $\zeta_{Y,p}\colon L_Y(y_p) \to k$ for each $p \in I$, and $\xi_{X,p}\colon M_X(x_p) \to k$ and $\xi_{Y,p}\colon M_Y(y_p) \to k$ for each $p \in J$. If $g_Xg_Y > 0$ and $I \cap J \neq \emptyset$, choose them such that

$$\{(\zeta_{X,p}^{\otimes\widetilde{\beta}} \otimes \xi_{X,p}^{\otimes\widetilde{\alpha}}, \zeta_{Y,p}^{\otimes\widetilde{\beta}} \otimes \xi_{Y,p}^{\otimes\widetilde{\alpha}}) \,|\, p \in I \cap J\}$$

patch $L_X^{\otimes\widetilde{\beta}} \otimes M_X^{\otimes\widetilde{\alpha}}$ and $L_Y^{\otimes\widetilde{\beta}} \otimes M_Y^{\otimes\widetilde{\alpha}}$ to the sheaf $K_{I\cap J}$ given by (8.2.2). Consider the corresponding evaluation maps,

$$e_X\colon H^0(L_X) \to k_I,\ e_Y\colon H^0(L_Y) \to k_I,\ f_X\colon H^0(M_X) \to k_J,\ f_Y\colon H^0(M_Y) \to k_J.$$

Let $V := \mathrm{Im}(e_Y)$ and $W := \mathrm{Im}(f_X)$. Let $h_X := \dim V$ and $h_Y := \dim W$. Let

$$G_X := \mathrm{Grass}_{h_X}(k_I) \quad \text{and} \quad G_Y := \mathrm{Grass}_{h_Y}(k_J).$$

Consider the natural actions of the tori $k_I^*$ and $k_J^*$ on $k_I$ and $k_J$, and their respective actions on $G_X$ and $G_Y$. Denote by $\mathbb{O}_V$ and $\mathbb{O}_W$ the orbits of $V$ and $W$ under these actions, and by $\psi_X\colon k_I^* \to \mathbb{O}_V$ and $\psi_Y\colon k_J^* \to \mathbb{O}_W$ the orbit maps. If $g_Xg_Y > 0$, let

$$T := \{(s,t) \in k_I^* \times k_J^* \,|\, s_p^{\widetilde{\beta}} = t_p^{\widetilde{\alpha}} \text{ for every } p \in I \cap J\}, \tag{8.3.1}$$

and denote by $\mathbb{O}$ the orbit of $(V, W)$ under the induced action of $T$ on $G_X \times G_Y$.

**Lemma 8.4.** *Preserve* 4.1–2 *and* 8.3. *Then the following three statements hold.*

1. *If* (4.3.2a) *holds and $\alpha \neq 0$, then $e_X$ induces a closed embedding $\iota_X\colon G_X \to \mathbb{G}_X$ such that $\iota_X(\mathbb{O}_V) = \mathbb{V}_{\mu,X}$.*
2. *If* (4.3.2b) *holds and $\beta \neq 0$, then $f_Y$ induces a closed embedding $\iota_Y\colon G_Y \to \mathbb{G}_Y$ such that $\iota_Y(\mathbb{O}_W) = \mathbb{V}_{\mu,Y}$.*
3. *If* (4.3.2a–b) *hold, $\alpha \neq 0$ and $\beta \neq 0$, then $\mathbb{V}_\mu = (\iota_X \times \iota_Y)(\mathbb{O})$, where $\iota_X$ and $\iota_Y$ are the embeddings mentioned in Statements 1 and 2.*

*Proof.* Let's prove Statement 1. Since $\alpha \neq 0$, also $g_Y > 0$, and $\alpha_p > 0$ for every $p \in I$ by Lemma 3.1. By Riemann–Roch, $h^0(L_X) = g + |\alpha| - g_Y$ and $e_X$ is surjective. Now, $e_Y$ is injective by (4.3.2a). So $h^0(L_X) - g = |I| - h_X$ by (5.1.5). Thus, taking inverse images by $e_X$ gives us a closed embedding $\iota_X\colon G_X \to \mathbb{G}_X$. Since $g_Y > 0$ and (4.3.2a) holds, the description

of $\mathbb{V}_{\mu,X}$ given in Theorem 8.2 applies. Comparing this description with that of $\mathbb{O}_V$ given in 8.3, we get $\iota_X(\mathbb{O}_V) = \mathbb{V}_{\mu,X}$. The proof of Statement 1 is complete. Statement 2 is proved analogously.

Let's prove now Statement 3. Since $\alpha$ and $\beta$ are non–zero, also $g_X$ and $g_Y$ are non–zero. Since (4.3.2a–b) hold as well, the description of $\mathbb{V}_\mu$ given in Theorem 8.2 applies. Comparing this description with that of $\mathbb{O}$ given in 8.3, we get $\mathbb{V}_\mu = (\iota_X \times \iota_Y)(\mathbb{O})$. Statement 3 is proved. $\square$

**Theorem 8.5.** *Preserve* 4.1–2. *Then the following three statements hold.*

1. *If* (4.3.2a) *holds, then* $\mathbb{V}_{\mu,X}$ *is locally closed in* $\mathbb{G}_X$ *and isomorphic to a torus. In addition,* $\dim \mathbb{V}_{\mu,X} = |I| - 1$ *if* $|\alpha| > g_Y$*; otherwise,* $\mathbb{V}_{\mu,X} = \{H^0(L_X)\}$.
2. *If* (4.3.2b) *holds, then* $\mathbb{V}_{\mu,Y}$ *is locally closed in* $\mathbb{G}_Y$ *and isomorphic to a torus. In addition,* $\dim \mathbb{V}_{\mu,Y} = |J| - 1$ *if* $|\beta| > g_X$*; otherwise,* $\mathbb{V}_{\mu,Y} = \{H^0(M_Y)\}$.
3. *If* (4.3.2a–b) *hold, then* $\mathbb{V}_\mu$ *is closed in* $\mathbb{V}_{\mu,X} \times \mathbb{V}_{\mu,Y}$ *and isomorphic to a torus. In addition,*

$$\dim \mathbb{V}_\mu = \begin{cases} 0 & \text{if } |\alpha| = g_Y \text{ and } |\beta| = g_X, \\ |I| - 1 & \text{if } |\alpha| > g_Y \text{ and } |\beta| = g_X, \\ |J| - 1 & \text{if } |\alpha| = g_Y \text{ and } |\beta| > g_X, \\ |I \cup J| - 2 & \text{if } |\alpha| > g_Y \text{ and } |\beta| > g_X \text{ and } I \cap J = \emptyset, \\ |I \cup J| - 1 & \text{if } |\alpha| > g_Y \text{ and } |\beta| > g_X \text{ and } I \cap J \neq \emptyset. \end{cases}$$

*Proof.* We shall use the set–up of 8.3. Let's prove Statement 1. Assume first that $|\alpha| = g_Y$. Then Theorem 5.2 says that $(H^0(L_X), L_X)$ is the limit canonical aspect with focus on $X$ of any regular smoothing of $\widetilde{C}$. Therefore $\mathbb{V}_{\mu,X} = \{H^0(L_X)\}$.

Assume now that $|\alpha| > g_Y$. Then $\alpha \neq 0$ and $g_Y > 0$. By (4.3.2a), all the Plücker coordinates of $V$ in $G_X$ are non–zero. In addition, $V \neq k_I^*$ because $|\alpha| > g_Y$. Since the Plücker coordinates of $V$ are non–zero, $\psi_X(t) = V$ if and only if $t^b = t^c$ for all subsets $b$ and $c$ of $I$ satisfying $|b| = |c| = h_X$. Since $h_X < |I|$, given $p, q \in I$ distinct, there is a subset $b \subseteq I$ with $|b| = h_X$ such that $p \in b$ but $q \notin b$. Letting $c := b - p + q$, we have $t^b = t^c$ if and only if $t_p = t_q$. It follows that the orbit map $\psi_X$ factors through an isomorphism $k_I^*/k^* \to \mathbb{O}_V$, where $k^*$ is viewed inside $k_I^*$ under the diagonal embedding. So $\mathbb{O}_V$ is isomorphic to a torus of dimension $|I| - 1$.

Since $\alpha \neq 0$, Lemma 8.4 says that $\mathbb{V}_{\mu,X}$ is the image of $\mathbb{O}_V$ under a closed embedding $G_X \to \mathbb{G}_X$. Thus $\mathbb{V}_{\mu,X}$ is locally closed in $\mathbb{G}_X$ and isomorphic to a torus of dimension $|I|-1$. The proof of Statement 1 is complete. Statement 2 follows by analogy.

Let's prove Statement 3. If $|\beta| = g_X$ then $\mathbb{V}_\mu = \mathbb{V}_{\mu,X} \times \{H^0(M_Y)\}$. If $|\alpha| = g_Y$ then $\mathbb{V}_\mu = \{H^0(L_X)\} \times \mathbb{V}_{\mu,Y}$. So Statement 3 follows from Statement 1 in the first case, and from Statement 2 in the second case.

Assume now that $|\alpha| > g_Y$ and $|\beta| > g_X$. Then $\alpha$, $\beta$, $g_X$ and $g_Y$ are non–zero. View $k^*$ inside $k_I^*$ and $k_J^*$ under the diagonal embeddings. Let $T' := T \cap (k^* \times k^*)$, where $T$ is given by (8.3.1). Since $\widetilde{\alpha}$ and $\widetilde{\beta}$ are coprime, $T$ is a subtorus of dimension $|I \cup J|$ of $k_I^* \times k_J^*$. In addition, $T'$ is a one–dimensional subtorus of $T$ unless $I \cap J = \emptyset$, in which case $T' = k^* \times k^*$. Now, $\mathbb{O} \subseteq \mathbb{O}_V \times \mathbb{O}_W$ and the orbit map $\psi\colon T \to \mathbb{O}$ is the restriction to $T$ of $\psi_X \times \psi_Y$. Since $\psi_X$ and $\psi_Y$ factor through isomorphisms $k_I^*/k^* \to \mathbb{O}_V$ and $k_J^*/k^* \to \mathbb{O}_W$, then $\psi$ factors

through an isomorphism $T/T' \to \mathbb{O}$. So $\mathbb{O}$ is closed in $\mathbb{O}_V \times \mathbb{O}_W$ and isomorphic to a torus. In addition, $\dim \mathbb{O} = |I \cup J| - 1$ unless $I \cap J = \emptyset$, in which case $\dim \mathbb{O} = |I \cup J| - 2$.

Since $\alpha$ and $\beta$ are non–zero, Lemma 8.4 says that $\mathbb{V}_\mu$ is the image of $\mathbb{O}$ under a closed embedding $G_X \times G_Y \to \mathbb{G}_X \times \mathbb{G}_Y$ that sends $\mathbb{O}_V \times \mathbb{O}_W$ to $\mathbb{V}_{\mu,X} \times \mathbb{V}_{\mu,Y}$. Thus $\mathbb{V}_\mu$ is closed in $\mathbb{V}_{\mu,X} \times \mathbb{V}_{\mu,Y}$ and isomorphic to a torus of the dimension prescribed in Statement 3. $\square$

**8.6.** *Extreme cases.* If (4.3.2a) holds, then

$$\mathbb{V}_{\mu,X} = \{V \in \mathbb{G}_X |\ V \supseteq H^0(L_X(-\sum_{p\in I} x_p)) \text{ but } V \not\supseteq H^0(L_X(x_q - \sum_{p\in I} x_p)) \text{ for any } q \in I\}$$

in case $|\alpha| = g_Y + 1$, and

$$\mathbb{V}_{\mu,X} = \{V \in \mathbb{G}_X |\ V \supseteq H^0(L_X(-\sum_{p\in I} x_p)) \text{ but } V \not\subseteq H^0(L_X(-x_q)) \text{ for any } q \in I\}$$

in case $|\alpha| = g_Y + |I| - 1$. We prove here the latter case; the former is proved analogously.

So, suppose $|\alpha| = g_Y + |I| - 1$. If $\alpha = 0$ then $\mathbb{V}_{\mu,X} = \{H^0(L_X)\}$ by Theorem 8.5, which agrees with the above description of $\mathbb{V}_{\mu,X}$ in the case at hand. Assume now that $\alpha \neq 0$. From now on we shall use the set-up of 8.3. By (4.3.2a), all the Plücker coordinates of $V$ in $G_X$ are non–zero. Moreover, $h_X = 1$. Then $\mathbb{O}_V$ parameterizes all one–dimensional subspaces $H \subseteq k_I$ such that the restriction map $H \to k_q$ is an isomorphism for each $q \in I$. Now, Lemma 8.4 says that $\mathbb{V}_{\mu,X}$ is the image of $\mathbb{O}_V$ under the closed embedding $\iota_X \colon G_X \to \mathbb{G}_X$ induced by $e_X$. The description of $\mathbb{V}_{\mu,X}$ given in the last paragraph follows easily now.

If (4.3.2b) holds, and either $|\beta| = g_X + 1$ or $|\beta| = g_X + |J| - 1$, then an analogous description holds for $\mathbb{V}_{\mu,Y}$.

**8.7.** *Projections.* Preserve 4.1–2. Using Theorem 8.2 we can describe $\mathbb{V}_{\mu,X}$ and $\mathbb{V}_{\mu,Y}$ in terms of central projections. In addition, we can describe $\mathbb{V}_\mu \subseteq \mathbb{V}_{\mu,X} \times \mathbb{V}_{\mu,Y}$ in terms of multi–linear algebra.

In fact, assume (4.3.2a) holds and $g_Y > 0$. Then the natural map $\phi_Y \colon Y \to \mathbb{P}(H^0(L_Y))$ is defined along $\{y_p \,|\, p \in I\}$, and the image $\{\phi_Y(y_p) \,|\, p \in I\}$ spans $\mathbb{P}(H^0(L_Y))$. Now, $\alpha_p > 0$ for each $p \in I$ by Lemma 3.1. By Riemann–Roch, the natural map $\phi_X \colon X \to \mathbb{P}(H^0(L_X))$ is defined along $\{x_p \,|\, p \in I\}$, and the set $\{\phi_X(x_p) \,|\, p \in I\}$ spans a projective subspace $\mathbb{P}_X \subseteq \mathbb{P}(H^0(L_X))$ of dimension $|I| - 1$. Let $T_X$ be the set of all linear maps $\mathbb{P}_X \to \mathbb{P}(H^0(L_Y))$ sending $\phi_X(x_p)$ to $\phi_Y(y_p)$ for each $p \in I$. Then $T_X$ is a torus of dimension $|I| - 1$. For each $t \in T_X$ let $\mathbb{P}_t \subseteq \mathbb{P}_X$ be the base locus of $t$, and let $(V_t, L_X)$ be the linear system given by projection with center $\mathbb{P}_t$. Then $V_t \in \mathbb{V}_{\mu,X}$ and the map $\lambda_X \colon T_X \to \mathbb{V}_{\mu,X}$ sending $t$ to $V_t$ is an isomorphism.

Now, assume (4.3.2b) holds and $g_X > 0$. The sheaves $M_X$ and $M_Y$ define natural maps $\psi_X \colon X \to \mathbb{P}(H^0(M_X))$ and $\psi_Y \colon Y \to \mathbb{P}(H^0(M_Y))$. As above, $\psi_Y$ is defined along $\{y_p \,|\, p \in J\}$, and the projective subspace $\mathbb{P}_Y \subseteq \mathbb{P}(H^0(M_Y))$ spanned by $\{\psi_Y(y_p) \,|\, p \in J\}$ has dimension $|J| - 1$. In addition, $\psi_X$ is defined along $\{x_p \,|\, p \in J\}$, and $\{\psi_X(x_p) \,|\, p \in J\}$ spans $\mathbb{P}(H^0(M_X))$. Let $T_Y$ be the set of all linear maps $\mathbb{P}_Y \to \mathbb{P}(H^0(M_X))$ sending $\psi_Y(y_p)$ to $\psi_X(x_p)$ for each $p \in J$. As above, we define an isomorphism $\lambda_Y \colon T_Y \to \mathbb{V}_{\mu,Y}$.

Finally, assume (4.3.2a–b) hold and $g_X g_Y > 0$. Note that $\mathbb{P}_X = \mathbb{P}(V_X)$ and $\mathbb{P}_Y = \mathbb{P}(V_Y)$, where

$$V_X := \frac{H^0(L_X)}{H^0(L_X(-\sum_{p\in I} x_p))} \quad \text{and} \quad V_Y := \frac{H^0(M_Y)}{H^0(M_Y(-\sum_{p\in J} y_p))}.$$

Combining a $\widetilde{\beta}$–tuple with an $\widetilde{\alpha}$–tuple embedding, and composing with a Segre map, embed $\mathbb{P}_X \times \mathbb{P}(H^0(M_X))$ in $\overline{\mathbb{P}}_X$ and $\mathbb{P}(H^0(L_Y)) \times \mathbb{P}_Y$ in $\overline{\mathbb{P}}_Y$, where

$$\overline{\mathbb{P}}_X := \mathbb{P}(\mathrm{Symm}_{\widetilde{\beta}}(V_X) \otimes \mathrm{Symm}_{\widetilde{\alpha}}(H^0(M_X))),$$
$$\overline{\mathbb{P}}_Y := \mathbb{P}(\mathrm{Symm}_{\widetilde{\beta}}(H^0(L_Y)) \otimes \mathrm{Symm}_{\widetilde{\alpha}}(V_Y)).$$

Let $C_{I\cap J}$ be the blow–up of $C$ along $\Delta - (I \cap J)$, and $K_{I\cap J}$ the invertible sheaf on $C_{I\cap J}$ given by (8.2.2). Let $\phi\colon C_{I\cap J} \to \mathbb{P}(H^0(K_{I\cap J}))$ be the natural map, and $\overline{\mathbb{P}}_C$ the projective subspace of $\mathbb{P}(H^0(K_{I\cap J}))$ spanned by $\phi(I \cap J)$. In other words, $\overline{\mathbb{P}}_C = \mathbb{P}(\overline{V})$, where

$$\overline{V} := \frac{H^0(K_{I\cap J})}{H^0(K_{I\cap J}(-\sum_{p\in I\cap J} p))}.$$

Now, $K_{I\cap J}$ restricts to $L_X^{\otimes\widetilde{\beta}} \otimes M_X^{\otimes\widetilde{\alpha}}$ on $X$ and $L_Y^{\otimes\widetilde{\beta}} \otimes M_Y^{\otimes\widetilde{\alpha}}$ on $Y$. Thus, lifting sections, we obtain well–defined maps,

$$\frac{H^0(L_X^{\otimes\widetilde{\beta}} \otimes M_X^{\otimes\widetilde{\alpha}})}{H^0(L_X^{\otimes\widetilde{\beta}} \otimes M_X^{\otimes\widetilde{\alpha}}(-\sum_{p\in I} x_p))} \longrightarrow \overline{V} \quad \text{and} \quad \frac{H^0(L_Y^{\otimes\widetilde{\beta}} \otimes M_Y^{\otimes\widetilde{\alpha}})}{H^0(L_Y^{\otimes\widetilde{\beta}} \otimes M_Y^{\otimes\widetilde{\alpha}}(-\sum_{p\in J} y_p))} \longrightarrow \overline{V},$$

from which we derive natural linear maps $\tau_X\colon \overline{\mathbb{P}}_C \to \overline{\mathbb{P}}_X$ and $\tau_Y\colon \overline{\mathbb{P}}_C \to \overline{\mathbb{P}}_Y$, respectively.

Each $s \in T_X$ induces a linear map $\zeta_s\colon \overline{\mathbb{P}}_X \to \overline{\mathbb{P}}$, where

$$\overline{\mathbb{P}} := \mathbb{P}(\mathrm{Symm}_{\widetilde{\beta}}(H^0(L_Y)) \otimes \mathrm{Symm}_{\widetilde{\alpha}}(H^0(M_X))).$$

Also, each $t \in T_Y$ induces a linear map $\xi_t\colon \overline{\mathbb{P}}_Y \to \overline{\mathbb{P}}$. Let

$$T := \{(s,t) \in T_X \times T_Y \,|\, \zeta_s\tau_X = \xi_t\tau_Y\}.$$

Then $(\lambda_X \times \lambda_Y)(T) = \mathbb{V}_\mu$.

## 9. Boundary of tori orbits on Grassmannians

**Lemma 9.1.** *Let $I$ be a finite set and let $V \subseteq k_I$ be a non–zero vector subspace. Let $h := \dim V$ and $G := \mathrm{Grass}_h(k_I)$. For each ordered tripartition $\mathbb{I} = (I', \overline{I}, I'')$ of $I$, let*

$$V_{\mathbb{I}} := k_{I'} + (k_{\overline{I}} \cap (V + k_{I''})). \tag{9.1.1}$$

*Let $\mathbb{O}$ denote the orbit of $V$ under the natural action of $k_I^*$ on $G$. If all Plücker coordinates of $V$ in $G$ are non–zero, then the closure $\overline{\mathbb{O}} \subseteq G$ is the union of the orbits of the subspaces $V_{\mathbb{I}} \subseteq k_I$ obtained from all ordered tripartitions $\mathbb{I} = (I', \overline{I}, I'')$ of $I$ satisfying $|I'| < h \leq |I - I''|$.*

*Proof.* Let $\mathcal{B} := \{b \subseteq I;\ |b| = h\}$. For each $b \in \mathcal{B}$ let $p_b$ denote the corresponding Plücker coordinate on $G$. In terms of Plücker coordinates, $s \in k_I^*$ acts on a point of $G$ with coordinates $(p_b \,|\, b \in \mathcal{B})$ by taking it to the point with coordinates $(s^b p_b \,|\, b \in \mathcal{B})$.

Let $(\widetilde{p}_b \,|\, b \in \mathcal{B})$ be Plücker coordinates of $V$ in $G$. By assumption, $\widetilde{p}_b \neq 0$ for every $b \in \mathcal{B}$. Let $Z \subseteq G$ be defined by the following equations on Plücker coordinates,

$$\widetilde{p}_{b_1}\widetilde{p}_{b_2}p_{b_3}p_{b_4} = \widetilde{p}_{b_3}\widetilde{p}_{b_4}p_{b_1}p_{b_2} \qquad \text{for all } b_1, \dots, b_4 \in \mathcal{B} \text{ with } b_1 + b_2 = b_3 + b_4. \tag{9.1.2}$$

It's clear that $\mathbb{O} \subseteq Z$.

Let $W \in G$ with Plücker coordinates $(p_b \,|\, b \in \mathcal{B})$. Let

$$\mathcal{B}_W := \{b \in \mathcal{B} \,|\, p_b \neq 0\}, \quad I'_W := \bigcap_{b\in\mathcal{B}_W} b, \quad I''_W := I - \bigcup_{b\in\mathcal{B}_W} b.$$

Note that $|I'_W| \leq h \leq |I - I''_W|$. Set $\overline{I}_W := I - (I'_W \cup I''_W)$.

We divide the proof in five steps.

*Step 1*: Let $W \in Z$ with Plücker coordinates $(p_b \,|\, b \in \mathcal{B})$. Then the following two conditions hold.

$$\mathcal{B}_W = \{b \in \mathcal{B} \,|\, I'_W \subseteq b \subseteq I - I''_W\} \tag{9.1.3a}$$

$$\text{There is } \overline{s} \in k^*_{\overline{I}_W} \text{ such that } p_{b_1}\widetilde{p}_{b_2} = \overline{s}^{(b_1-b_2)}\widetilde{p}_{b_1}p_{b_2} \text{ for all } b_1, b_2 \in \mathcal{B}_W. \tag{9.1.3b}$$

Indeed, let's check (9.1.3a) first. If $b \in \mathcal{B}_W$ then $I'_W \subseteq b \subseteq I - I''_W$ by definition.

Conversely, suppose first that $|\mathcal{B}_W| = 1$. Then $I'_W = I - I''_W$, and thus (9.1.3a) holds. Suppose now that $|\mathcal{B}_W| > 1$. Let $b \in \mathcal{B}_W$. Then $I'_W \neq b \neq I - I''_W$. Let $i, j \in I$ such that $i \in b - I'_W$ and $j \in I - I''_W - b$. Using an induction argument, we need only show that $b - i + j \in \mathcal{B}_W$. Since $i \notin I'_W$ and $j \notin I''_W$, there are $b_1, b_2 \in \mathcal{B}_W$ such that $i \notin b_1$ and $j \in b_2$. We'll find $b_0 \in \mathcal{B}_W$ such that $i \notin b_0$ and $j \in b_0$. If $j \in b_1$ or $i \notin b_2$, set $b_0 := b_1$ or $b_0 := b_2$, respectively. Suppose that $j \notin b_1$ and $i \in b_2$. Then $|b_1 \cap b_2| < h$, and hence there is $u \in b_1 - b_2$. Set $b_0 := b_1 - u + j$. Then $i \notin b_0$ and $j \in b_0$. Since $b_1 + b_2 = b_0 + (b_2 - j + u)$, Equations (9.1.2) imply that $b_0 \in \mathcal{B}_W$. Hence, since $b + b_0 = (b - i + j) + (b_0 - j + i)$, Equations (9.1.2) imply that $b - i + j \in \mathcal{B}_W$ as well. Thus (9.1.3a) holds.

Let's now check (9.1.3b). If $\overline{I}_W = \emptyset$ then $|\mathcal{B}_W| = 1$, and hence (9.1.3b) holds trivially. Assume that $\overline{I}_W \neq \emptyset$, and fix $\overline{i} \in \overline{I}_W$. Define $\overline{s} \in k^*_{\overline{I}_W}$ by letting $\overline{s}_{\overline{i}} := 1$ and

$$\overline{s}_i := \frac{p_{b-\overline{i}+i}\widetilde{p}_b}{\widetilde{p}_{b-\overline{i}+i}p_b}$$

for each $i \in \overline{I}_W - \overline{i}$, where $b \in \mathcal{B}_W$ is chosen such that $i \notin b$ and $\overline{i} \in b$. Observe that such $b$ exists because (9.1.3a) holds. Moreover, $\overline{s}_i$ is independent of the choice of $b$, because if $b' \in \mathcal{B}_W$ is such that $i \notin b'$ and $\overline{i} \in b'$, then $b + (b' - \overline{i} + i) = (b - \overline{i} + i) + b'$, and hence the claimed independence of $\overline{s}_i$ follows from Equations (9.1.2).

If $i$ and $j$ are distinct elements of $\overline{I}_W$, then

$$p_b\widetilde{p}_{b-j+i} = (\overline{s}_j/\overline{s}_i)\widetilde{p}_b p_{b-j+i} \tag{9.1.4}$$

for every $b \in \mathcal{B}_W$ such that $i \notin b$ and $j \in b$. Indeed, if either $\overline{i} = i$ or $\overline{i} = j$, then (9.1.4) holds by definition of $\overline{s}_j$ or $\overline{s}_i$. Suppose that $\overline{i} \in \overline{I}_W - i - j$. If $\overline{i} \in b$, then

$$\frac{\overline{s}_i}{\overline{s}_j} = \Big(\frac{p_{b-\overline{i}+i}\widetilde{p}_b}{\widetilde{p}_{b-\overline{i}+i}p_b}\Big)/\Big(\frac{p_{b-\overline{i}+i}\widetilde{p}_{b-j+i}}{\widetilde{p}_{b-\overline{i}+i}p_{b-j+i}}\Big) = \frac{p_{b-j+i}\widetilde{p}_b}{p_b\widetilde{p}_{b-j+i}}.$$

If $\overline{i} \notin b$, then

$$\frac{\overline{s}_i}{\overline{s}_j} = \Big(\frac{p_{b-j+i}\widetilde{p}_{b-j+\overline{i}}}{\widetilde{p}_{b-j+i}p_{b-j+\overline{i}}}\Big)/\Big(\frac{p_b\widetilde{p}_{b-j+\overline{i}}}{\widetilde{p}_b p_{b-j+\overline{i}}}\Big) = \frac{p_{b-j+i}\widetilde{p}_b}{p_b\widetilde{p}_{b-j+i}}.$$

Either way, (9.1.4) holds. Now, (9.1.3b) follows from (9.1.4) and an induction argument. Step 1 is complete.

*Step 2*: Let $W \in G$ with Plücker coordinates $(p_b \,|\, b \in \mathcal{B})$. If (9.1.3a,b) hold, then $W \in \overline{\mathbb{O}}$. Indeed, let $\overline{s} \in k^*_{\overline{I}_W}$ as in (9.1.3b). Define $u \in \mathbb{Z}_I$ and $s \in k^*_I$ by letting for each $i \in I$,

$$u_i := \begin{cases} -1 & \text{if } i \in I'_W, \\ 0 & \text{if } i \in \overline{I}_W, \\ 1 & \text{if } i \in I''_W. \end{cases} \quad \text{and} \quad s_i := \begin{cases} 1 & \text{if } i \in I'_W, \\ \overline{s}_i & \text{if } i \in \overline{I}_W, \\ 1 & \text{if } i \in I''_W. \end{cases}$$

Define a map $\zeta\colon k^* \to k_I^*$ by letting $\zeta(r)_i := s_i r^{u_i}$ for each $i \in I$ and each $r \in k^*$. For each $r \in k^*$ let $W_r := \zeta(r)V$ and $(p_b(r) \,|\, b \in \mathcal{B})$ be Plücker coordinates of $W_r$. Because of our choice of $\zeta$, the power of $r$ in the expression for $p_b(r)$ is minimum exactly among those $b \in \mathcal{B}_W$. Let $\overline{W} \in G$ be the limit of $W_r$ as $r$ goes to 0, and $(\overline{p}_b \,|\, b \in \mathcal{B})$ Plücker coordinates of $\overline{W}$. Of course, $\overline{W} \in \overline{\mathbb{O}}$. Now, $\overline{p}_b \neq 0$ if and only if $b \in \mathcal{B}_W$. Moreover, if $b_1, b_2 \in \mathcal{B}_W$ then

$$\frac{\overline{p}_{b_1}}{\overline{p}_{b_2}} = s^{(b_1-b_2)}\frac{\widetilde{p}_{b_1}}{\widetilde{p}_{b_2}} = \overline{s}^{(b_1-b_2)}\frac{\widetilde{p}_{b_1}}{\widetilde{p}_{b_2}} = \frac{p_{b_1}}{p_{b_2}},$$

where the last equality holds by (9.1.3b). So $\overline{W} = W$, and hence $W \in \overline{\mathbb{O}}$. Step 2 is complete.

*Step 3*: If $\mathbb{I} = (I, \overline{I}, I'')$ is an ordered tripartition of $I$ such that $|I'| \leq h \leq |I - I''|$, then $\dim V_{\mathbb{I}} = h$ and $(p_b \,|\, b \in \mathcal{B})$ are Plücker coordinates of $V_{\mathbb{I}}$, where

$$p_b := \begin{cases} \widetilde{p}_b & \text{if } I' \subseteq b \subseteq I - I'', \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, let $v_1, \dots, v_h$ be a basis of $V$ in $k_I$. Write $I = \{1, \dots, \delta\}$ with

$$\begin{aligned} I' &= \{1, \dots, |I'|\}, \\ \overline{I} &= \{|I'|+1, \dots, |I' + \overline{I}|\}, \\ I'' &= \{|I' + \overline{I}|+1, \dots, |I|\}. \end{aligned}$$

Since no Plücker coordinate of $V$ is zero, we may assume that the matrix $(v_{i,j})_{1\leq i,j\leq h}$ is the identity. Since $|I'| \leq h \leq |I - I''|$, a basis of $V_{\mathbb{I}}$ is given by the vectors $\overline{v}_1, \dots, \overline{v}_h$ defined by

$$\overline{v}_{i,j} := \begin{cases} 0 & \text{if } j \in I'', \\ 0 & \text{if } i \in I' \text{ and } j \in \overline{I}, \\ v_{i,j} & \text{otherwise.} \end{cases}$$

In particular, $\dim V_{\mathbb{I}} = h$. It follows as well that $(p_b \,|\, b \in \mathcal{B})$ are Plücker coordinates of $V_{\mathbb{I}}$. Step 3 is complete.

*Step 4*: If $W \in \overline{\mathbb{O}}$, then there is an ordered tripartition $\mathbb{I} = (I', \overline{I}, I'')$ of $I$ such that $|I'| < h \leq |I - I''|$, and such that $W$ is in the orbit of $V_{\mathbb{I}}$ defined by (9.1.1). Indeed, let $W \in \overline{\mathbb{O}}$ with Plücker coordinates $(p_b \,|\, b \in \mathcal{B})$. Since $\mathbb{O} \subseteq Z$, we have $W \in Z$, and hence (9.1.3a,b) hold by Step 1. Since (9.1.3b) holds, we may replace $W$ by a certain point in its orbit and assume that $p_b = \widetilde{p}_b$ for every $b \in \mathcal{B}_W$. Hence, using Step 3, it follows from (9.1.3a) that $W = V_{\widetilde{\mathbb{I}}}$, where $\widetilde{\mathbb{I}} := (I'_W, \overline{I}_W, I''_W)$.

If $|I'_W| < h$ set $\mathbb{I} := \widetilde{\mathbb{I}}$. Step 4 is complete in this case. If $|I'_W| = h$ then $\overline{I}_W = \emptyset$, and thus $V_{\widetilde{\mathbb{I}}} = k_{I'_W}$. In addition, $I'_W \neq \emptyset$ because $h > 0$. Let $\overline{I} \subseteq I'_W$ be any non–empty subset and put $\mathbb{I} := (I'_W - \overline{I}, \overline{I}, I''_W)$. Now, $V_{\mathbb{I}} = k_{I'_W}$ since $h = |I - I''_W|$. Then $V_{\mathbb{I}} = V_{\widetilde{\mathbb{I}}} = W$. Step 4 is complete.

*Step 5*: If $\mathbb{I} = (I', \overline{I}, I'')$ is an ordered tripartition of $I$ such that $|I'| < h \leq |I - I''|$, then $V_{\mathbb{I}} \in \overline{\mathbb{O}}$. Indeed, let $(p_b \,|\, b \in \mathcal{B})$ be Plücker coordinates of $V_{\mathbb{I}}$. By Step 3,

$$\mathcal{B}_{V_{\mathbb{I}}} = \{b \in \mathcal{B} \,|\, I' \subseteq b \subseteq I - I''\}.$$

If $h < |I - I''|$ then $I'_{V_{\mathbb{I}}} = I'$ and $I''_{V_{\mathbb{I}}} = I''$. If $h = |I - I''|$ then $I'_{V_{\mathbb{I}}} = I' \cup \overline{I}$ and $I''_{V_{\mathbb{I}}} = I''$. In either case (9.1.3a) holds. By Step 3 as well, (9.1.3b) holds trivially with $\overline{s} = 1$. By Step 2, $V_{\mathbb{I}} \in \overline{\mathbb{O}}$, thus finishing Step 5.

Steps 4 and 5 show the lemma. ☐

**Lemma 9.2.** *Let $\Delta$ be a finite set, $I$ and $J$ non–empty subsets of $\Delta$. Let $V \subseteq k_I$ and $W \subseteq k_J$ be non–zero vector subspaces. Let $h_1 := \dim V$ and $G_1 := \mathrm{Grass}_{h_1}(k_I)$. Let $h_2 := \dim W$ and $G_2 := \mathrm{Grass}_{h_2}(k_J)$. For each ordered tripartition $\mathbb{I} = (I', \overline{I}, I'')$ of $I$ and each ordered tripartition $\mathbb{J} = (J', \overline{J}, J'')$ of $J$, let*

$$V_{\mathbb{I}} := k_{I'} + (k_{\overline{I}} \cap (V + k_{I''})) \quad \text{and} \quad W_{\mathbb{J}} := k_{J'} + (k_{\overline{J}} \cap (W + k_{J''})). \tag{9.2.1}$$

*Let $\lambda$ and $\tau$ be positive integers, and set*

$$T := \{(s,t) \in k_I^* \times k_J^* \,|\, s_i^\tau t_j^\lambda = s_j^\tau t_i^\lambda \text{ for all } i, j \in I \cap J\}.$$

*Let $\mathbb{O}$ denote the orbit of $(V, W)$ under the induced action of $T$ on $G_1 \times G_2$. If all Plücker coordinates of $V$ and $W$ in $G_1$ and $G_2$ are non–zero, then the closure $\overline{\mathbb{O}} \subseteq G_1 \times G_2$ is the union of the orbits of the pairs $(V_{\mathbb{I}}, W_{\mathbb{J}})$ obtained from all ordered tripartitions $\mathbb{I} = (I', \overline{I}, I'')$ of $I$ and $\mathbb{J} = (J', \overline{J}, J'')$ of $J$ satisfying*

$$|I'| < h_1 \leq |I - I''| \quad \text{and} \quad |J'| < h_2 \leq |J - J''|, \tag{9.2.2}$$

*and such that the following two conditions hold.*

$$I' \cap J \not\subseteq J' \cap I \text{ or } J'' \cap I \not\subseteq I'' \cap J \Longrightarrow J \cap (I - I') \subseteq J'' \cap I, \tag{9.2.3a}$$

$$J' \cap I \not\subseteq I' \cap J \text{ or } I'' \cap J \not\subseteq J'' \cap I \Longrightarrow I \cap (J - J') \subseteq I'' \cap J. \tag{9.2.3b}$$

*Proof.* Let

$$\mathcal{B} := \{b \subseteq I;\ |b| = h_1\} \quad \text{and} \quad \mathcal{C} := \{c \subseteq J;\ |c| = h_2\}.$$

For each $b \in \mathcal{B}$ let $p_b$ denote the corresponding Plücker coordinate on $G_1$. For each $c \in \mathcal{C}$ let $q_c$ denote the corresponding Plücker coordinate on $G_2$.

Let $\widetilde{\nu} := (V, W)$. Let $(\widetilde{p}_b \,|\, b \in \mathcal{B})$ be Plücker coordinates of $V$ in $G_1$ and $(\widetilde{q}_c \,|\, c \in \mathcal{C})$ be Plücker coordinates of $W$ in $G_2$. By assumption, $\widetilde{p}_b \neq 0$ for every $b \in \mathcal{B}$ and $\widetilde{q}_c \neq 0$ for every $c \in \mathcal{C}$. Let $Z \subseteq G_1 \times G_2$ be defined by the following equations on Plücker coordinates:

$$\widetilde{p}_{b_1}\widetilde{p}_{b_2}p_{b_3}p_{b_4} = \widetilde{p}_{b_3}\widetilde{p}_{b_4}p_{b_1}p_{b_2} \quad \text{for all } b_1, \dots, b_4 \in \mathcal{B} \text{ with } b_1 + b_2 = b_3 + b_4, \tag{9.2.4a}$$

$$\widetilde{q}_{c_1}\widetilde{q}_{c_2}q_{c_3}q_{c_4} = \widetilde{q}_{c_3}\widetilde{q}_{c_4}q_{c_1}q_{c_2} \quad \text{for all } c_1, \dots, c_4 \in \mathcal{C} \text{ with } c_1 + c_2 = c_3 + c_4, \tag{9.2.4b}$$

$$\widetilde{p}_{b_1}^\tau\widetilde{q}_{c_1}^\lambda p_{b_2}^\tau q_{c_2}^\lambda = \widetilde{p}_{b_2}^\tau\widetilde{q}_{c_2}^\lambda p_{b_1}^\tau q_{c_1}^\lambda \quad \text{for all } b_1, b_2 \in \mathcal{B} \text{ and } c_1, c_2 \in \mathcal{C} \text{ satisfying (9.2.5)}, \tag{9.2.4c}$$

where

$$\begin{cases} b_1 + c_1 = b_2 + c_2, \\ b_1 \cap (I \setminus J) = b_2 \cap (I \setminus J), \\ c_1 \cap (J \setminus I) = c_2 \cap (J \setminus I). \end{cases} \tag{9.2.5}$$

Given $\nu \in G_1 \times G_2$ with Plücker coordinates $(p_b \,|\, b \in \mathcal{B})$ and $(q_c \,|\, c \in \mathcal{C})$, let

$$\mathcal{B}_\nu := \{b \in \mathcal{B} \,|\, p_b \neq 0\} \quad \text{and} \quad \mathcal{C}_\nu := \{c \in \mathcal{C} \,|\, q_c \neq 0\}.$$

Put

$$I'_\nu := \bigcap_{b \in \mathcal{B}_\nu} b, \quad I''_\nu := I - \bigcup_{b \in \mathcal{B}_\nu} b, \quad J'_\nu := \bigcap_{c \in \mathcal{C}_\nu} c, \quad J''_\nu := J - \bigcup_{c \in \mathcal{C}_\nu} c.$$

Note that $|I'_\nu| \leq h_1 \leq |I - I''_\nu|$ and $|J'_\nu| \leq h_2 \leq |J - J''_\nu|$. Set

$$\overline{I}_\nu := I - (I'_\nu \cup I''_\nu) \quad \text{and} \quad \overline{J}_\nu := J - (J'_\nu \cup J''_\nu),$$

and let

$$\overline{T}_\nu := \{(s,t)\in k^*_{\overline{I}_\nu}\times k^*_{\overline{J}_\nu}\,|\, s_i^\tau t_j^\lambda = s_j^\tau t_i^\lambda \text{ for all } i,j\in \overline{I}_\nu\cap\overline{J}_\nu\}.$$

We divide the proof in five steps.

*Step 1*: If $\nu\in\mathbb{O}$ then $\nu\in Z$. Indeed, let $\nu\in\mathbb{O}$ with Plücker coordinates $(p_b\,|\,b\in\mathcal{B})$ and $(q_c\,|\,c\in\mathcal{C})$. Since $\nu\in\mathbb{O}$, there is $(s,t)\in T$ such that $p_b = s^b\widetilde{p}_b$ for every $b\in\mathcal{B}$ and $q_c = t^c\widetilde{q}_c$ for every $c\in\mathcal{C}$. Clearly, (9.2.4a–b) hold. As for (9.2.4c), it holds if and only if $s^{\tau(b_1-b_2)} = t^{\lambda(c_2-c_1)}$ for all $b_1,b_2\in\mathcal{B}$ and $c_1,c_2\in\mathcal{C}$ satisfying (9.2.5). Since $(s,t)\in T$, we need only show that

$$(b_1\cap J)-(b_2\cap J) = b_1-b_2 = c_2-c_1 = (c_2\cap I)-(c_1\cap I).$$

But the three equalities above hold because (9.2.5) holds. So $\nu\in Z$, finishing Step 1.

*Step 2*: Let $\nu\in Z$ with Plücker coordinates $(p_b\,|\,b\in\mathcal{B})$ and $(q_c\,|\,c\in\mathcal{C})$. Then the following four conditions hold.

(9.2.6a) $\quad \mathcal{B}_\nu = \{b\in\mathcal{B}\,|\,I'_\nu\subseteq b\subseteq I-I''_\nu\}$,

(9.2.6b) $\quad \mathcal{C}_\nu = \{c\in\mathcal{C}\,|\,J'_\nu\subseteq c\subseteq J-J''_\nu\}$,

(9.2.6c) $$\begin{cases} I'_\nu\cap J\not\subseteq J'_\nu\cap I \text{ or } J''_\nu\cap I\not\subseteq I''_\nu\cap J \Longrightarrow J\cap(I-I'_\nu)\subseteq J''_\nu\cap I,\\ J'_\nu\cap I\not\subseteq I'_\nu\cap J \text{ or } I''_\nu\cap J\not\subseteq J''_\nu\cap I \Longrightarrow I\cap(J-J'_\nu)\subseteq I''_\nu\cap J.\end{cases}$$

(9.2.6d) $\quad$ There is $(\overline{s},\overline{t})\in\overline{T}_\nu$ such that $$\begin{cases} p_{b_1}\widetilde{p}_{b_2} = \overline{s}^{(b_1-b_2)}\widetilde{p}_{b_1}p_{b_2} & \text{for all } b_1,b_2\in\mathcal{B}_\nu,\\ q_{c_1}\widetilde{q}_{c_2} = \overline{t}^{(c_1-c_2)}\widetilde{q}_{c_1}q_{c_2} & \text{for all } c_1,c_2\in\mathcal{C}_\nu.\end{cases}$$

Indeed, (9.2.6a–b) follow as in the proof of Lemma 9.1.

Let's check (9.2.6c) now. Suppose there are $i\in I'_\nu\cap(J-J'_\nu)$ and $j\in(J-J''_\nu)\cap(I-I'_\nu)$. Since $i\in I'_\nu$ and $j\not\in I'_\nu$, there is $b\in\mathcal{B}_\nu$ such that $i\in b$ and $j\not\in b$. Since $i\not\in J'_\nu$ and $j\not\in J''_\nu$, and since (9.2.6b) holds, there is $c\in\mathcal{C}_\nu$ such that $i\not\in c$ and $j\in c$. Since $b+c=(b-i+j)+(c-j+i)$, Equations (9.2.4c) imply that $b-i+j\in\mathcal{B}_\nu$. However, $i\in I'_\nu$, reaching a contradiction. Analogously, if $J'_\nu\cap(I-I'_\nu)\neq\emptyset$ then $I\cap(J-J'_\nu)\subseteq I''_\nu\cap J$.

Suppose now that there are $i\in J''_\nu\cap(I-I''_\nu)$ and $j\in(J-J''_\nu)\cap(I-I'_\nu)$. Since $i\not\in I''_\nu$ and $j\not\in I'_\nu$, and since (9.2.6a) holds, there is $b\in\mathcal{B}_\nu$ such that $i\in b$ and $j\not\in b$. Since $i\in J''_\nu$ and $j\not\in J''_\nu$, there is $c\in\mathcal{C}_\nu$ such that $i\not\in c$ and $j\in c$. Since $b+c=(b-i+j)+(c-j+i)$, it follows from Equations (9.2.4c) that $c-j+i\in\mathcal{C}_\nu$. However, $i\in J''_\nu$, reaching a contradiction. Analogously, if $I''_\nu\cap(J-J''_\nu)\neq\emptyset$ then $I\cap(J-J'_\nu)\subseteq I''_\nu\cap J$. So (9.2.6c) is proved.

Let's check (9.2.6d). As in the proof of Lemma 9.1, there are $\overline{s}\in k^*_{\overline{I}_\nu}$ and $\overline{t}\in k^*_{\overline{J}_\nu}$ such that

(9.2.7) $$\begin{cases} p_{b_1}\widetilde{p}_{b_2} = \overline{s}^{(b_1-b_2)}\widetilde{p}_{b_1}p_{b_2} & \text{for all } b_1,b_2\in\mathcal{B}_\nu,\\ q_{c_1}\widetilde{q}_{c_2} = \overline{t}^{(c_1-c_2)}\widetilde{q}_{c_1}q_{c_2} & \text{for all } c_1,c_2\in\mathcal{C}_\nu.\end{cases}$$

If $i$ and $j$ are distinct elements of $\overline{I}_\nu\cap\overline{J}_\nu$, let $b\in\mathcal{B}_\nu$ and $c\in\mathcal{C}_\nu$ such that $i\not\in b\cup c$ and $j\in b\cap c$. Since $b+(c-j+i)=(b-j+i)+c$, Equations (9.2.4c) and (9.2.7) imply that $\overline{s}_i^\tau\overline{t}_j^\lambda = \overline{s}_j^\tau\overline{t}_i^\lambda$. So $(\overline{s},\overline{t})\in\overline{T}_\nu$, and hence (9.2.6d) holds. Step 2 is complete.

*Step 3*: Let $\nu\in G_1\times G_2$ with Plücker coordinates $(p_b\,|\,b\in\mathcal{B})$ and $(q_c\,|\,c\in\mathcal{C})$. If (9.2.6a–d) hold, then $\nu\in\overline{\mathbb{O}}$. Indeed, for $u\in\mathbb{Z}_I$, $v\in\mathbb{Z}_J$, $s\in k^*_I$ and $t\in k^*_J$, define the map

$$\zeta := (\zeta_1,\zeta_2)\colon k^*\longrightarrow k^*_I\times k^*_J,$$

by letting $\zeta_1(r)_i := s_i r^{u_i}$ for each $i \in I$ and $\zeta_2(r)_j := t_j r^{v_j}$ for each $j \in J$. We'll choose $u$, $v$, $s$ and $t$ as indicated below.

Let $(\overline{s}, \overline{t}) \in \overline{T}_\nu$ as in (9.2.6d). For each $i \in \overline{I}_\nu$ set $u_i := 0$ and $s_i := \overline{s}_i$. For each $j \in \overline{J}_\nu$ set $v_j := 0$ and $t_j := \overline{t}_j$.

For each $i \in (I - \overline{I}_\nu) \setminus J$ and each $j \in (J - \overline{J}_\nu) \setminus I$ set $s_i := 1$ and $t_j := 1$, and put

$$u_i := \begin{cases} -1 & \text{if } i \in I'_\nu, \\ 1 & \text{if } i \in I''_\nu, \end{cases} \quad \text{and} \quad v_j := \begin{cases} -1 & \text{if } j \in J'_\nu, \\ 1 & \text{if } j \in J''_\nu. \end{cases}$$

Choose the remaining $s_i$ and $t_j$ in such a way that $(s, t) \in T$. As for the remaining $u_i$ and $v_j$ there are three cases to consider.

*Case 1*: If

$$I \cap J = (I'_\nu \cap J'_\nu) \cup (\overline{I}_\nu \cap \overline{J}_\nu) \cup (I''_\nu \cap J''_\nu), \tag{9.2.8}$$

set $u_i := -\lambda$ and $v_i := -\tau$ if $i \in I'_\nu \cap J'_\nu$ and $u_i := \lambda$ and $v_i := \tau$ if $i \in I''_\nu \cap J''_\nu$.

*Case 2*: In case (9.2.8) does not hold, but

$$I \cap J = (I'_\nu \cap J'_\nu) \cup (\overline{I}_\nu \cap J'_\nu) \cup (I''_\nu \cap J'_\nu) \cup (I''_\nu \cap \overline{J}_\nu) \cup (I''_\nu \cap J''_\nu), \tag{9.2.9}$$

set

$$u_i := \begin{cases} -\lambda & \text{if } i \in I'_\nu \cap J'_\nu, \\ \lambda & \text{if } i \in I''_\nu \cap J'_\nu, \\ 2\lambda & \text{if } i \in I''_\nu \cap \overline{J}_\nu, \\ 3\lambda & \text{if } i \in I''_\nu \cap J''_\nu, \end{cases} \quad \text{and} \quad v_i := \begin{cases} -3\tau & \text{if } i \in J'_\nu \cap I'_\nu, \\ -2\tau & \text{if } i \in J'_\nu \cap \overline{I}_\nu, \\ -\tau & \text{if } i \in J'_\nu \cap I''_\nu, \\ \tau & \text{if } i \in J''_\nu \cap I''_\nu. \end{cases} \tag{9.2.10}$$

*Case 3*: In case (9.2.8) does not hold, but

$$I \cap J = (I'_\nu \cap J'_\nu) \cup (I'_\nu \cap \overline{J}_\nu) \cup (I'_\nu \cap J''_\nu) \cup (\overline{I}_\nu \cap J''_\nu) \cup (I''_\nu \cap J''_\nu), \tag{9.2.11}$$

set $u_i$ and $v_i$ as in (9.2.10), but exchanging $I$ with $J$, $\lambda$ with $\tau$, and the $u_i$ with the $v_i$.

If (9.2.9) and (9.2.11) hold, then (9.2.8) holds as well. So the three cases above are independent. Moreover, one of the three cases occur. Indeed, if $J \cap (I - I'_\nu) \not\subseteq J''_\nu$ and $I \cap (J - J'_\nu) \not\subseteq I''_\nu$, then (9.2.8) occurs by (9.2.6c). On the other hand, if $J \cap (I - I'_\nu) \subseteq J''_\nu$ then (9.2.11) holds, and if $I \cap (J - J'_\nu) \subseteq I''_\nu$, then (9.2.9) holds.

We chose $\zeta$ in such a way that it factors through $T$. Indeed, since $(s, t) \in T$, it is enough to check that $\tau(u_i - u_j) = \lambda(v_i - v_j)$ for all $i, j \in I \cap J$, what can easily be done in each of the three cases above.

Let $\nu_r := \zeta(r)\widetilde{\nu}$ for each $r \in k^*$, and let $(p_b(r) \,|\, b \in \mathcal{B})$ and $(q_c(r) \,|\, c \in \mathcal{C})$ be Plücker coordinates of $\nu_r$. Because of our choice of $\zeta$, the power of $r$ in the expression for $p_b(r)$ is minimum exactly among those $b \in \mathcal{B}_\nu$. Likewise, the power of $r$ in the expression for $q_c(r)$ is minimum exactly among those $c \in \mathcal{C}_\nu$. Let $\overline{\nu} \in G_1 \times G_2$ be the limit of $\nu_r$ as $r$ goes to 0. As in the proof of Lemma 9.1, we have $\overline{\nu} = \nu$. Then $\nu \in \overline{\mathbb{O}}$, finishing Step 3.

*Step 4*: Let $\nu \in \overline{\mathbb{O}}$. Then there are ordered tripartitions $\mathbb{I} = (I, \overline{I}, I'')$ and $\mathbb{J} = (J', \overline{J}, J'')$ of $I$ and $J$, respectively, such that (9.2.2–3) hold, and such that $\nu$ is in the orbit of $(V_{\mathbb{I}}, W_{\mathbb{J}})$ defined by (9.2.1). Indeed, let $(p_b \,|\, b \in \mathcal{B})$ and $(q_c \,|\, c \in \mathcal{C})$ be the Plücker coordinates of $\nu \in \overline{\mathbb{O}}$. By Step 1, $\nu \in Z$. Hence (9.2.6a–d) hold by Step 2. Since (9.2.6d) holds, we may replace $\nu$ by a certain point in its orbit and assume that $p_b = \widetilde{p}_b$ for every $b \in \mathcal{B}_\nu$ and $q_c = \widetilde{q}_c$ for every $c \in \mathcal{C}_\nu$. Let $\widetilde{\mathbb{I}} := (I'_\nu, \overline{I}_\nu, I''_\nu)$ and $\widetilde{\mathbb{J}} := (J'_\nu, \overline{J}_\nu, J''_\nu)$. As in the proof of Lemma 9.1, we have $\nu = (V_{\widetilde{\mathbb{I}}}, W_{\widetilde{\mathbb{J}}})$.

There are four cases to consider.

*Case 1*: If $|I'_\nu| < h_1$ and $|J'_\nu| < h_2$, set $\mathbb{I} := \widetilde{\mathbb{I}}$ and $\mathbb{J} := \widetilde{\mathbb{J}}$. Clearly (9.2.2) holds, whereas (9.2.3a–b) hold by (9.2.6c). Since $\nu = (V_{\mathbb{I}}, W_{\mathbb{J}})$, Step 4 is complete for Case 1.

*Case 2*: If $|I'_\nu| = h_1$ and $|J'_\nu| < h_2$, then set $\mathbb{I} := (I'_\nu - \overline{I}, \overline{I}, I''_\nu)$ for a certain non–empty subset $\overline{I} \subseteq I'_\nu$ to be defined below. Set $\mathbb{J} := \widetilde{\mathbb{J}}$. As in the proof of Lemma 9.1, we have $\nu = (V_{\mathbb{I}}, W_{\mathbb{J}})$.

Clearly (9.2.2) holds. We need only choose $\overline{I}$ such that (9.2.3a–b) hold, and we'll do it according to whether (9.2.8), (9.2.9) or (9.2.11) occurs. If (9.2.8) or (9.2.9) occurs, set $\overline{I} := I'_\nu$. If (9.2.8) does not occur but (9.2.11) does, set

$$\overline{I} := \begin{cases} I'_\nu \setminus J & \text{if } I'_\nu \not\subseteq J, \\ I'_\nu \cap J''_\nu & \text{if } I'_\nu \subseteq J \text{ and } I'_\nu \not\subseteq (J'_\nu \cup \overline{J}_\nu), \\ I'_\nu \cap \overline{J}_\nu & \text{if } I'_\nu \subseteq (J'_\nu \cup \overline{J}_\nu) \text{ and } I'_\nu \not\subseteq J'_\nu, \\ I'_\nu & \text{if } I'_\nu \subseteq J'_\nu. \end{cases}$$

With any of the above choices (9.2.3a–b) hold for $\mathbb{I}$ and $\mathbb{J}$.

*Case 3*: If $|I'_\nu| < h_1$ and $|J'_\nu| = h_2$, proceed as in Case 2, exchanging $I$ with $J$.

*Case 4*: If $|I'_\nu| = h_1$ and $|J'_\nu| = h_2$, let $\overline{I} \subseteq I'_\nu$ and $\overline{J} \subseteq J'_\nu$ be non–empty subsets, and put

$$\mathbb{I} := (I'_\nu - \overline{I}, \overline{I}, I''_\nu) \quad \text{and} \quad \mathbb{J} := (J'_\nu - \overline{J}, \overline{J}, J''_\nu).$$

As in Case 2, we need only choose $\overline{I}$ and $\overline{J}$ such that (9.2.3a–b) hold. If (9.2.8) occurs, set $\overline{I} := I'_\nu$ and $\overline{J} := J'_\nu$. If (9.2.9) occurs but (9.2.8) does not, set $\overline{I} := I'_\nu$ and $\overline{J} := I''_\nu \cap J'_\nu$. If (9.2.11) occurs but (9.2.8) does not, set $\overline{I} := I'_\nu \cap J''_\nu$ and $\overline{J} := J'_\nu$. With any of the above choices (9.2.3a–b) hold for $\mathbb{I}$ and $\mathbb{J}$. Step 4 is complete.

*Step 5*: If $\nu = (V_{\mathbb{I}}, W_{\mathbb{J}})$, where $\mathbb{I} = (I', \overline{I}, I'')$ and $\mathbb{J} = (J', \overline{J}, J'')$ are ordered tripartitions of $I$ and $J$ such that (9.2.2–3) hold, then $\nu \in \overline{\mathbb{O}}$. Indeed, let $(p_b \,|\, b \in \mathcal{B})$ and $(q_c \,|\, c \in \mathcal{C})$ be Plücker coordinates of $\nu$. As in the proof of Lemma 9.1,

$$\mathcal{B}_\nu = \{b \in \mathcal{B} | I' \subseteq b \subseteq I - I''\}.$$

If $h_1 < |I - I''|$ then $I'_\nu = I'$ and $I''_\nu = I''$. If $h_1 = |I - I''|$ then $I'_\nu = I' \cup \overline{I}$ and $I''_\nu = I''$. In either case (9.2.6a) holds. Analogously, (9.2.6b) holds. Now, (9.2.6c) is simply a restatement of (9.2.3) if $h_1 < |I - I''|$ and $h_2 < |J - J''|$. In the remaining cases, (9.2.6c) can be obtained from (9.2.3) as well. Finally, as in the proof of Lemma 9.1, Condition (9.2.6d) holds trivially with $(\overline{s}, \overline{t}) = 1$. By Step 3, $\nu \in \overline{\mathbb{O}}$, thus finishing Step 5.

Steps 4 and 5 show the lemma. □

## 10. The variety of limit canonical systems

**Theorem 10.1.** *Preserve* 4.1 *and assume* (4.3.1). *Let* $\mu, \mu' \in \mathbb{Z}^+_\Delta$. *Then the following five assertions hold.*

1. $\mathbb{V}_{\mu,X} = \mathbb{V}_{\mu',X}$ *if and only if* $\alpha_\mu = \alpha_{\mu'}$ *and either* $I_\mu = I_{\mu'}$ *or* $|\alpha_\mu| = g_Y$.
2. $\mathbb{V}_{\mu,Y} = \mathbb{V}_{\mu',Y}$ *if and only if* $\beta_\mu = \beta_{\mu'}$ *and either* $J_\mu = J_{\mu'}$ *or* $|\beta_\mu| = g_X$.
3. $\mathbb{V}_\mu = \mathbb{V}_{\mu'}$ *if and only if* $\mathbb{V}_{\mu,X} = \mathbb{V}_{\mu',X}$ *and* $\mathbb{V}_{\mu,Y} = \mathbb{V}_{\mu',Y}$.
4. *Either* $\mathbb{V}_\mu = \mathbb{V}_{\mu'}$ *or* $\mathbb{V}_\mu \cap \mathbb{V}_{\mu'} = \emptyset$.
5. *If* $\mu' = t\mu$ *for a certain* $t \in \mathbb{Z}^+$, *then* $\mathbb{V}_\mu = \mathbb{V}_{\mu'}$.

*Proof.* We take two preliminary steps.

*Step 1*: If $\mathbb{V}_{\mu,X} \cap \mathbb{V}_{\mu',X} \neq \emptyset$, then $\alpha_\mu = \alpha_{\mu'}$ and either $I_\mu = I_{\mu'}$ or $|\alpha_\mu| = g_Y$. Indeed, let $V \in \mathbb{G}_X$, and assume that $V \in \mathbb{V}_{\mu,X} \cap \mathbb{V}_{\mu',X}$. Let

$$L_X := \omega_X(\textstyle\sum_{p\in\Delta}(1+\alpha_{\mu,p})x_p) \quad \text{and} \quad L_Y := \omega_Y(\textstyle\sum_{p\in I_\mu} y_p - \sum_{p\in\Delta}\alpha_{\mu,p}y_p).$$

Then $V \subseteq H^0(L_X)$. Now, $x_p$ is not a base point of $(V, L_X)$ for any $p \in \Delta$ by Theorem 5.2. So $g_Y - \alpha_{\mu,p}$ is the minimum order of $(V, \mathbb{L})$ at $x_p$ for each $p \in \Delta$. Analogously, $g_Y - \alpha_{\mu',p}$ is the minimum order of $(V, \mathbb{L})$ at $x_p$ for each $p \in \Delta$. So $\alpha_\mu = \alpha_{\mu'}$.

Assume now that $|\alpha_\mu| > g_Y$. Let $F \subseteq \Delta$. We claim that

$$V \supseteq H^0(L_X(-\textstyle\sum_{p\in F} x_p)) \tag{10.1.1}$$

if and only if $F \supseteq I_\mu$. In fact, $V \supseteq H^0(L_X(-\sum_{p\in I_\mu} x_p))$ by Theorem 5.2. So, if $F \supseteq I_\mu$ then (10.1.1) holds. Conversely, suppose (10.1.1) holds. Since $\alpha_\mu \neq 0$, by Riemann–Roch, $\{x_p \,|\, p \in \Delta\}$ imposes independent conditions on $H^0(L_X)$. So, since $V \supseteq H^0(L_X(-\sum_{p\in I_\mu} x_p))$ by Theorem 5.2, it follows from (10.1.1) that $V \supseteq H^0(L_X(-\sum_{p\in F'} x_p))$, where $F' := F \cap I_\mu$. We may thus assume that $F' = F$.

Suppose by contradiction that $F \neq I_\mu$. Replacing $F$ by a larger subset, if necessary, we may assume $|I_\mu - F| = 1$. Since $\{x_p \,|\, p \in \Delta\}$ imposes independent conditions on $H^0(L_X)$, there is $s \in H^0(L_X)$ such that $s(x_p) = 0$ for every $p \in F$ and $s(x_p) \neq 0$ for $p \in I_\mu - F$. So $s \in V$ by (10.1.1). Let $C_{I_\mu}$ be the blow–up of $C$ along $\Delta - I_\mu$. By Theorem 8.2, there is an invertible sheaf $L$ on $C_{I_\mu}$ such that $L|_X \cong L_X$ and $L|_Y \cong L_Y$, and such that $V$ is the image of the restriction map $H^0(L) \to H^0(L_X)$. Let $\widetilde{s} \in H^0(L)$ lifting $s$ and $t := \widetilde{s}|_Y$. So $t \in H^0(L_Y(-\sum_{p\in F} y_p))$. Since $|I_\mu - F| = 1$ and $|\alpha| > g_Y$, it follows from (4.3.2a) that $h^0(L_Y(-\sum_{p\in F} y_p)) = 0$. So $t = 0$, and hence $s(x_p) = 0$ for every $p \in \Delta$. The contradiction proves our claim.

Now, $\alpha_\mu = \alpha_{\mu'}$. By analogy, (10.1.1) holds if and only if $F \supseteq I_{\mu'}$. So $I_\mu = I_{\mu'}$. Step 1 is complete.

*Step 2*: Assume that $\alpha_\mu = \alpha_{\mu'}$ and either $I_\mu = I_{\mu'}$ or $|\alpha_\mu| = g_Y$. Then $\mathbb{V}_{\mu,X} = \mathbb{V}_{\mu',X}$. Indeed, it follows from Theorem 8.2 and the hypothesis that $\mathbb{V}_{\mu,X} = \mathbb{V}_{\mu',X}$ if $|\alpha_\mu| > g_Y$. Now, if $|\alpha_\mu| = g_Y$ then

$$\mathbb{V}_{\mu',X} = \mathbb{V}_{\mu,X} = \{H^0(L_X)\}$$

by Theorem 8.5. Step 2 is complete.

Steps 1 and 2 prove Statement 1. Statement 2 follows by analogy. Let's prove Statement 3. By definition, $\mathbb{V}_{\mu,X}$ and $\mathbb{V}_{\mu,Y}$ are the images of $\mathbb{V}_\mu$ in $\mathbb{G}_X$ and $\mathbb{G}_Y$, respectively, for each $\mu \in \mathbb{Z}^\Delta_+$. Hence, if $\mathbb{V}_\mu = \mathbb{V}_{\mu'}$ then $\mathbb{V}_{\mu,X} = \mathbb{V}_{\mu',X}$ and $\mathbb{V}_{\mu,Y} = \mathbb{V}_{\mu',Y}$.

Conversely, assume that $\mathbb{V}_{\mu,X} = \mathbb{V}_{\mu',X}$ and $\mathbb{V}_{\mu,Y} = \mathbb{V}_{\mu',Y}$. By Statements 1 and 2, we have $(\alpha_\mu, \beta_\mu) = (\alpha_{\mu'}, \beta_{\mu'})$. In addition, $I_\mu = I_{\mu'}$ when $|\alpha_\mu| > g_Y$ and $J_\mu = J_{\mu'}$ when $|\beta_\mu| > g_X$. So, if $|\alpha_\mu| > g_Y$ and $|\beta_\mu| > g_X$ then $\mathbb{V}_\mu = \mathbb{V}_{\mu'}$ by Theorem 8.2. Now, if $|\alpha_\mu| = g_Y$ or $|\beta_\mu| = g_X$ then

$$\mathbb{V}_\mu = \mathbb{V}_{\mu,X} \times \mathbb{V}_{\mu,Y} = \mathbb{V}_{\mu',X} \times \mathbb{V}_{\mu',Y} = \mathbb{V}_{\mu'},$$

where the first and last equalities follow from Theorem 8.5. Statement 3 is proved.

Let's prove Statement 4. Suppose that $\mathbb{V}_\mu \cap \mathbb{V}_{\mu'} \neq \emptyset$. Then $\mathbb{V}_{\mu,X} \cap \mathbb{V}_{\mu',X} \neq \emptyset$ as well. Thus $\mathbb{V}_{\mu,X} = \mathbb{V}_{\mu',X}$ by Step 1. Analogously, $\mathbb{V}_{\mu,Y} = \mathbb{V}_{\mu',Y}$. By Statement 3, $\mathbb{V}_\mu = \mathbb{V}_{\mu'}$. Statement 4 is proved.

Finally, let's prove Statement 5. Suppose $\mu' = t\mu$ for a certain $t \in \mathbb{Z}^+$. Then the numerical data $(\alpha_\mu, I_\mu)$ associated with $\mu$ and $g_Y$ is equal to that associated with $\mu'$ and $g_Y$. Analogously, the numerical data $(\beta_\mu, J_\mu)$ associated with $\mu$ and $g_X$ is equal to that associated with $\mu'$ and $g_X$. Thus $\mathbb{V}_\mu = \mathbb{V}_{\mu'}$ by Statements 1, 2 and 3. □

**Definition 10.2.** Preserve 4.1 and assume (4.3.1). For each $\mu \in \mathbb{Q}^+_\Delta$ let

$$\mathbb{V}_\mu := \mathbb{V}_{t\mu}, \quad \mathbb{V}_{\mu,X} := \mathbb{V}_{t\mu,X}, \quad \mathbb{V}_{\mu,Y} := \mathbb{V}_{t\mu,Y},$$

where $t$ is any positive integer such that $t\mu \in \mathbb{Z}^+_\Delta$. (By Theorem 10.1, $\mathbb{V}_\mu$ is well defined.) Give $\mathbb{Q}^+_\Delta$ the topology induced by the Euclidean topology of $\mathbb{R}_\Delta$. Let

$$\mathbb{V} := \bigcup_{\mu \in \mathbb{Q}^+_\Delta} \mathbb{V}_\mu \subseteq \mathbb{G}.$$

We call $\mathbb{V}$ the *variety of limit canonical systems of $C$*. We call each $\mathbb{V}_\mu$ a *stratum* of $\mathbb{V}$.

**Proposition 10.3.** *Preserve* 4.1 *and assume* (4.3.1)*. For each $\mu \in \mathbb{Q}^+_\Delta$ let*

$$\mathbb{U}_\mu := \{\mu' \in \mathbb{Q}^+_\Delta \,|\, \mathbb{V}_{\mu'} = \mathbb{V}_\mu\}.$$

*Then each $\mathbb{U}_\mu$ is convex and homogeneous, and the covering $\{\mathbb{U}_\mu \,|\, \mu \in \mathbb{Q}^+_\Delta\}$ of $\mathbb{Q}^+_\Delta$ is finite.*

*Proof.* By Theorem 10.1, $\mathbb{V}_{t\mu} = \mathbb{V}_\mu$ for each $\mu \in \mathbb{Q}^+_\Delta$ and each $t \in \mathbb{Q}^+$. In other words, each $\mathbb{U}_\mu$ is homogeneous.

Let $\mu, \mu' \in \mathbb{Q}^+_\Delta$. Clearly, $\mathbb{U}_\mu = \mathbb{U}_{\mu'}$ if and only if $\mathbb{V}_\mu = \mathbb{V}_{\mu'}$. So, by Theorem 10.1, $\mathbb{U}_\mu = \mathbb{U}_{\mu'}$ if and only if $(\alpha_\mu, \beta_\mu) = (\alpha_{\mu'}, \beta_{\mu'})$,

$$I_\mu = I_{\mu'} \text{ when } |\alpha_\mu| > g_Y \quad \text{and} \quad J_\mu = J_{\mu'} \text{ when } |\beta_\mu| > g_X. \tag{10.3.1}$$

On the other hand, $0 \leq \alpha_{\mu,p} \leq g_Y$ and $0 \leq \beta_{\mu,p} \leq g_X$ for each $p \in \Delta$ and $I_\mu, J_\mu \subseteq \Delta$. So, as $\mu$ runs over $\mathbb{Q}^+_\Delta$, the associated numerical data $\alpha_\mu, \beta_\mu, I_\mu, J_\mu$ runs over a finite set. Therefore, the covering $\{\mathbb{U}_\mu \,|\, \mu \in \mathbb{Q}^+_\Delta\}$ of $\mathbb{Q}^+_\Delta$ is finite.

Assume now that $\mathbb{U}_\mu = \mathbb{U}_{\mu'}$. Let $t \in [0,1] \cap \mathbb{Q}$. Set

$$\mu(t) := t\mu + (1-t)\mu', \quad \rho(t) := t\rho_\mu + (1-t)\rho_{\mu'}, \quad \sigma(t) := t\sigma_\mu + (1-t)\sigma_{\mu'},$$

and put

$$\begin{aligned} I(t) :=& \{p \in \Delta \,|\, \rho(t)_p \geq \rho(t)_q \text{ for every } q \in \Delta\}, \\ J(t) :=& \{p \in \Delta \,|\, \sigma(t)_p \leq \sigma(t)_q \text{ for every } q \in \Delta\}. \end{aligned}$$

Then $I(t) = I_\mu \cap I_{\mu'}$ if $I_\mu \cap I_{\mu'} \neq \emptyset$ and $J(t) = J_\mu \cap J_{\mu'}$ if $J_\mu \cap J_{\mu'} \neq \emptyset$. By (10.3.1),

$$I(t) = I_\mu \text{ when } |\alpha_\mu| > g_Y \quad \text{and} \quad J(t) = J_\mu \text{ when } |\beta_\mu| > g_X. \tag{10.3.2}$$

Now, since $(\alpha_\mu, \beta_\mu) = (\alpha_{\mu'}, \beta_{\mu'})$, it follows that

$$(\alpha_{\mu(t)}, \beta_{\mu(t)}) = (\alpha_\mu, \beta_\mu) \quad \text{and} \quad (I_{\mu(t)}, J_{\mu(t)}) = (I(t), J(t)).$$

Then $\mathbb{V}_{\mu(t)} = \mathbb{V}_\mu$ by (10.3.2) and Theorem 10.1. So $\mathbb{U}_\mu$ is convex. □

**Lemma 10.4.** *Preserve* 4.1*. Then, for each $\mu \in \mathbb{Q}^+_\Delta$ there exists an open neighbourhood $U_\mu \subseteq \mathbb{Q}^+_\Delta$ of $\mu$ such that, for each open neighbourhood $U \subseteq U_\mu$ of $\mu$, the following four statements hold.*

1. *If $\overline{\mu} \in U$ then there are unique ordered tripartitions $(I', \overline{I}, I'')$ of $I_\mu$ and $(J', \overline{J}, J'')$ of $J_\mu$ satisfying the following six properties.*

$$g_Y + |I''| \leq |\alpha_\mu| < g_Y + |I_\mu - I'|, \tag{10.4.1a}$$

$$g_X + |J''| \leq |\beta_\mu| < g_X + |J_\mu - J'|, \tag{10.4.1b}$$

$$\overline{I} = I_{\overline{\mu}}, \tag{10.4.1c}$$

$$\alpha_{\overline{\mu},p} = \alpha_{\mu,p} - 1 \textit{ if } p \in I'' \textit{ and } \alpha_{\overline{\mu},p} = \alpha_{\mu,p} \textit{ otherwise,} \tag{10.4.1d}$$

$$\overline{J} = J_{\overline{\mu}}, \tag{10.4.1e}$$

$$\beta_{\overline{\mu},p} = \beta_{\mu,p} - 1 \textit{ if } p \in J'' \textit{ and } \beta_{\overline{\mu},p} = \beta_{\mu,p} \textit{ otherwise.} \tag{10.4.1f}$$

*If $g_X g_Y > 0$, then $(I', \overline{I}, I'')$ and $(J', \overline{J}, J'')$ satisfy the two properties below as well:*

$$I' \cap J_\mu \not\subseteq J' \cap I_\mu \textit{ or } J'' \cap I_\mu \not\subseteq I'' \cap J_\mu \Longrightarrow J_\mu \cap (I_\mu - I') \subseteq J'' \cap I_\mu, \tag{10.4.1g}$$

$$J' \cap I_\mu \not\subseteq I' \cap J_\mu \textit{ or } I'' \cap J_\mu \not\subseteq J'' \cap I_\mu \Longrightarrow I_\mu \cap (J_\mu - J') \subseteq I'' \cap J_\mu. \tag{10.4.1h}$$

2. *If $g_Y > 0$ and $(I', \overline{I}, I'')$ is an ordered tripartition of $I_\mu$ satisfying* (10.4.1a)*, then there is $\overline{\mu} \in U$ satisfying* (10.4.1c,d)*.*
3. *If $g_X > 0$ and $(J', \overline{J}, J'')$ is an ordered tripartition of $J_\mu$ satisfying* (10.4.1b)*, then there is $\overline{\mu} \in U$ satisfying* (10.4.1e,f)*.*
4. *If $g_X g_Y > 0$ and $(I', \overline{I}, I'')$ and $(J', \overline{J}, J'')$ are ordered tripartitions of $I_\mu$ and $J_\mu$ satisfying* (10.4.1a,b,g,h)*, then there is $\overline{\mu} \in U$ satisfying* (10.4.1c–f)*.*

*Proof.* To ease notation, let

$$(\alpha, \rho, I, \beta, \sigma, J) := (\alpha_\mu, \rho_\mu, I_\mu, \beta_\mu, \sigma_\mu, J_\mu).$$

Let $U_\mu \subseteq \mathbb{Q}_\Delta$ consist of the points of the form $\mu + \varepsilon$ for all $\varepsilon \in \mathbb{Q}_\Delta$ satisfying

$$|\varepsilon_p| < \frac{\min(\min^*(\rho_q, \mu_q - \rho_q, \sigma_q, \mu_q - \sigma_q) \mid q \in \Delta)}{3(1 + \max(\max(\alpha_q, \beta_q) \mid q \in \Delta))} \quad \text{for every } p \in \Delta, \tag{10.4.2}$$

where "min$^*$" means the smallest non–zero number among the four listed. If (10.4.2) holds, then $|\varepsilon_p| < \mu_p$ for every $p \in \Delta$, and hence $U_\mu \subseteq \mathbb{Q}_\Delta^+$.

We take three preliminary steps.

*Step 1*: For each $\varepsilon \in \mathbb{Q}_\Delta$ there are ordered tripartitions $(I', \overline{I}, I'')$ of $I$ and $(J', \overline{J}, J'')$ of $J$ such that (10.4.1a,b) and (10.4.3a,b) below hold.

$$\varepsilon_p \alpha_p < \varepsilon_q \alpha_q = \varepsilon_r \alpha_r < \varepsilon_s \alpha_s \text{ for all } p \in I', \text{ all } q, r \in \overline{I} \text{ and all } s \in I'', \tag{10.4.3a}$$

$$\varepsilon_p \beta_p < \varepsilon_q \beta_q = \varepsilon_r \beta_r < \varepsilon_s \beta_s \text{ for all } p \in J', \text{ all } q, r \in \overline{J} \text{ and all } s \in J''. \tag{10.4.3b}$$

Indeed, for each $\tau \in \mathbb{Q}$ let

$$I'_\tau := \{p \in I \mid \varepsilon_p \alpha_p < \tau\}, \quad \overline{I}_\tau := \{p \in I \mid \varepsilon_p \alpha_p = \tau\}, \quad I''_\tau := \{p \in I \mid \varepsilon_p \alpha_p > \tau\}.$$

Since $g_Y \leq |\alpha| < g_Y + |I|$, there is a unique $\lambda_\varepsilon \in \mathbb{Q}$ such that

$$g_Y + |I''_{\lambda_\varepsilon}| \leq |\alpha| < g_Y + |I - I'_{\lambda_\varepsilon}|.$$

Let $(I', \overline{I}, I'') := (I'_{\lambda_\varepsilon}, \overline{I}_{\lambda_\varepsilon}, I''_{\lambda_\varepsilon})$. Then (10.4.1a) and (10.4.3a) hold. By analogy, we construct $(J', \overline{J}, J'')$ satisfying (10.4.1b) and (10.4.3b). Step 1 is complete.

*Step 2*: If $g_X g_Y > 0$ and there are ordered tripartitions $(I', \overline{I}, I'')$ of $I$ and $(J', \overline{J}, J'')$ of $J$ satisfying (10.4.3) for a certain $\varepsilon \in \mathbb{Q}_\Delta$, then (10.4.1g,h) hold. Indeed, if $p, q \in I \cap J$ then $\mu_p \alpha_p = \mu_q \alpha_q$ and $\mu_p \beta_p = \mu_q \beta_q$. Hence

$$\alpha_p \beta_q = \alpha_q \beta_p \text{ for all } p, q \in I \cap J. \tag{10.4.4}$$

Assume that $g_X g_Y > 0$. Let $(I', \overline{I}, I'')$ and $(J', \overline{J}, J'')$ be ordered tripartitions of $I$ and $J$ satisfying (10.4.3) for a certain $\varepsilon \in \mathbb{Q}_\Delta$. If there are $p \in I' \cap (J - J')$ and $q \in (I - I') \cap (J - J'')$, then $\varepsilon_p \alpha_p < \varepsilon_q \alpha_q$ and $\varepsilon_q \beta_q \leq \varepsilon_p \beta_p$ by (10.4.3). Now, $\alpha_p \beta_p > 0$ because $g_X g_Y > 0$. Using (10.4.4) we get

$$\varepsilon_p \alpha_p \beta_p < \varepsilon_q \alpha_q \beta_p = \varepsilon_q \beta_q \alpha_p \leq \varepsilon_p \alpha_p \beta_p,$$

reaching a contradiction. If there are $p \in J'' \cap (I - I'')$ and $q \in (I - I') \cap (J - J'')$, then $\varepsilon_p \alpha_p \leq \varepsilon_q \alpha_q$ and $\varepsilon_q \beta_q < \varepsilon_p \beta_p$ by (10.4.3). Using (10.4.4) as before, we get $\varepsilon_p \alpha_p \beta_p < \varepsilon_p \alpha_p \beta_p$, reaching a contradiction. So (10.4.1g) holds. By analogy, (10.4.1h) holds. Step 2 is complete.

*Step 3*: Let $\overline{\mu} := \mu + \varepsilon$ for $\varepsilon \in \mathbb{Q}_\Delta$ such that (10.4.2) holds. Let $(I', \overline{I}, I'')$ and $(J', \overline{J}, J'')$ be ordered tripartitions of $I$ and $J$ such that (10.4.1a,b) hold. If (10.4.3a) holds, then so do (10.4.1c,d). If (10.4.3b) holds, then so do (10.4.1e,f). Indeed, define $\overline{\alpha}, \overline{\beta} \in \mathbb{Z}_\Delta$ by letting for each $p \in \Delta$,

$$\overline{\alpha}_p := \begin{cases} \alpha_p - 1 & \text{if } p \in I'', \\ \alpha_p & \text{otherwise,} \end{cases} \quad \text{and} \quad \overline{\beta}_p := \begin{cases} \beta_p - 1 & \text{if } p \in J'', \\ \beta_p & \text{otherwise.} \end{cases}$$

By (10.4.1a,b),

$$g_Y \leq |\overline{\alpha}| < g_Y + |\overline{I}| \quad \text{and} \quad g_X \leq |\overline{\beta}| < g_X + |\overline{J}|.$$

If (10.4.3a) holds, let $\lambda := \varepsilon_p \alpha_p$ for (any) $p \in \overline{I}$, and define $\overline{\rho} \in \mathbb{Q}_\Delta$ by letting for each $p \in \Delta$,

$$\overline{\rho}_p := \begin{cases} \overline{\mu}_p + \varepsilon_p \alpha_p - \lambda & \text{if } p \in I', \\ \overline{\mu}_p & \text{if } p \in \overline{I}, \\ \varepsilon_p \alpha_p - \lambda & \text{if } p \in I'', \\ \rho_p + \varepsilon_p(1 + \alpha_p) - \lambda & \text{if } p \notin I. \end{cases}$$

Then

$$\overline{\mu}_p(\overline{\alpha}_p + 1) - \overline{\rho}_p = \overline{\mu}_q(\overline{\alpha}_q + 1) - \overline{\rho}_q \quad \text{for all } p, q \in \Delta.$$

In addition, it follows from (10.4.2) and (10.4.3a) that $0 < \overline{\rho}_p \leq \overline{\mu}_p$ for every $p \in \Delta$, with $\overline{\rho}_p = \overline{\mu}_p$ if and only if $p \in \overline{I}$. So $(\overline{\alpha}, \overline{\rho}, \overline{I}) = (\alpha_{\overline{\mu}}, \rho_{\overline{\mu}}, I_{\overline{\mu}})$, proving (10.4.1c,d).

If (10.4.3b) holds, let $\tau := \varepsilon_p \beta_p$ for (any) $p \in \overline{J}$, and define $\overline{\sigma} \in \mathbb{Q}_\Delta$ by letting for each $p \in \Delta$,

$$\overline{\sigma}_p := \begin{cases} \tau - \varepsilon_p \beta_p & \text{if } p \in J', \\ 0 & \text{if } p \in \overline{J}, \\ \overline{\mu}_p + \tau - \varepsilon_p \beta_p & \text{if } p \in J'', \\ \sigma_p + \tau - \varepsilon_p \beta_p & \text{if } p \notin J. \end{cases}$$

As before, $(\overline{\beta}, \overline{\sigma}, \overline{J}) = (\beta_{\overline{\mu}}, \sigma_{\overline{\mu}}, J_{\overline{\mu}})$, proving (10.4.1e,f). Step 3 is complete.

Let's prove that $U := U_\mu$ satisfies Statement 1. Indeed, let $\overline{\mu} := \mu + \varepsilon$ for $\varepsilon \in \mathbb{Q}_\Delta$ such that (10.4.2) holds. By Step 1, there are ordered tripartitions $(I', \overline{I}, I'')$ of $I$ and $(J', \overline{J}, J'')$ of $J$ such that (10.4.1a,b) and (10.4.3a,b) hold. So (10.4.1c–f) hold by Step 3. If $g_X g_Y > 0$ then (10.4.1g,h) hold as well by Step 2. The uniqueness of the tripartitions follows from (10.4.1c–f).

Let $U \subseteq U_\mu$ be an open neighbourhood of $\mu$. Let's prove Statement 2. Assume $g_Y > 0$, and let $(I', \overline{I}, I'')$ be an ordered tripartition of $I$ satisfying (10.4.1a). Define $\upsilon \in \mathbb{Q}_\Delta$ by letting for each $p \in \Delta$,

$$\upsilon_p := \begin{cases} 0 & \text{if } p \in I' \cup (\Delta - I), \\ \mu_p & \text{if } p \in \overline{I}, \\ 2\mu_p & \text{if } p \in I''. \end{cases} \tag{10.4.5}$$

Let $\varepsilon := t\upsilon$, where $t \in \mathbb{Q}^+$ is such that $\overline{\mu} := \mu + \varepsilon$ lies in $U$. Now, $\alpha_p > 0$ for each $p \in I$ because $g_Y > 0$. Then (10.4.3a) follows from (10.4.5). So (10.4.1c,d) hold by Step 3. Analogously, Statement 3 holds.

Let's prove Statement 4. Assume that $g_X g_Y > 0$. Let $(I', \overline{I}, I'')$ and $(J', \overline{J}, J'')$ be ordered tripartitions of $I$ and $J$ satisfying (10.4.1a,b,g,h). Define $\upsilon \in \mathbb{Q}_\Delta$ according to the following three cases.

*Case 1*: If

$$I \cap J = (I' \cap J') \cup (\overline{I} \cap \overline{J}) \cup (I'' \cap J''), \tag{10.4.6}$$

let for each $p \in \Delta$,

$$\upsilon_p := \begin{cases} 0 & \text{if } p \in I' \cup J' \cup (\Delta - (I \cup J)), \\ \mu_p & \text{if } p \in \overline{I} \cup \overline{J}, \\ 2\mu_p & \text{if } p \in I'' \cup J''. \end{cases}$$

*Case 2*: If (10.4.6) does not hold, but

$$I \cap J = (I' \cap J') \cup (\overline{I} \cap J') \cup (I'' \cap J') \cup (I'' \cap \overline{J}) \cup (I'' \cap J''),$$

let for each $p \in \Delta$,

$$\upsilon_p := \begin{cases} 0 & \text{if } p \in I' \cup (J' \setminus I) \cup (\Delta - (I \cup J)), \\ \mu_p & \text{if } p \in \overline{I}, \\ 2\mu_p & \text{if } p \in I'' \cap J', \\ 3\mu_p & \text{if } p \in \overline{J}, \\ 4\mu_p & \text{if } p \in J'' \cup (I'' \setminus J). \end{cases}$$

*Case 3*: If (10.4.6) does not hold, but

$$I \cap J = (I' \cap J') \cup (I' \cap \overline{J}) \cup (I' \cap J'') \cup (\overline{I} \cap J'') \cup (I'' \cap J''),$$

define $\upsilon_p$ for each $p \in \Delta$ as in Case 2, but now exchanging $I$ with $J$.

As in the proof of Lemma 9.2, it follows from (10.4.1g,h) that the three cases above are independent, and one of them occurs.

Let $\varepsilon := t\upsilon$, where $t \in \mathbb{Q}^+$ is such that $\overline{\mu} := \mu + \varepsilon$ lies in $U$. Now, $\alpha_p > 0$ for each $p \in I$ and $\beta_p > 0$ for each $p \in J$ because $g_X g_Y > 0$. Then Conditions (10.4.3a,b) hold in each of the three cases above. So (10.4.1c–f) hold by Step 3. $\square$

**Theorem 10.5.** *Preserve* 4.1 *and assume* (4.3.1). *Then, for each $\mu \in \mathbb{Q}^+_\Delta$, there is an open neighbourhood $U_\mu \subseteq \mathbb{Q}^+_\Delta$ of $\mu$ such that the closure $\overline{\mathbb{V}}_\mu \subseteq \mathbb{G}$ satisfies*

$$\overline{\mathbb{V}}_\mu = \bigcup_{\overline{\mu} \in U} \mathbb{V}_{\overline{\mu}}$$

*for every open neighbourhood $U \subseteq U_\mu$ of $\mu$.*

*Proof.* Let $\mu \in \mathbb{Q}^+_\Delta$. By Theorem 10.1, we may assume that $\mu \in \mathbb{Z}^+_\Delta$. Let $U_\mu$ be the open neighborhood of $\mu$ given by Lemma 10.4. To ease notation, let

$$(\alpha, \rho, I, \beta, \sigma, J) := (\alpha_\mu, \rho_\mu, I_\mu, \beta_\mu, \sigma_\mu, J_\mu).$$

In addition, if $g_X g_Y > 0$ let $(\widetilde{\alpha}, \widetilde{\beta}) := (\widetilde{\alpha}_\mu, \widetilde{\beta}_\mu)$. Let $L_X$, $L_Y$, $M_X$ and $M_Y$ be the invertible sheaves given by (4.2.1). As in 8.3, let $C_{I\cap J}$ denote the blow–up of $C$ along $\Delta - (I \cap J)$. Fix isomorphisms $\zeta_{X,p} \colon L_X(x_p) \to k$ and $\zeta_{Y,p} \colon L_Y(y_p) \to k$ for each $p \in I$, and $\xi_{X,p} \colon M_X(x_p) \to k$ and $\xi_{Y,p} \colon M_Y(y_p) \to k$ for each $p \in J$. If $g_X g_Y > 0$ and $I \cap J \neq \emptyset$, choose them such that

$$\{(\zeta_{X,p}^{\otimes\widetilde{\beta}} \otimes \xi_{X,p}^{\otimes\widetilde{\alpha}}, \zeta_{Y,p}^{\otimes\widetilde{\beta}} \otimes \xi_{Y,p}^{\otimes\widetilde{\alpha}}) \,|\, p \in I \cap J\}$$

patch $L_X^{\otimes\widetilde{\beta}} \otimes M_X^{\otimes\widetilde{\alpha}}$ and $L_Y^{\otimes\widetilde{\beta}} \otimes M_Y^{\otimes\widetilde{\alpha}}$ to the sheaf $K_{I\cap J}$ given by (8.2.2). Consider the corresponding evaluation maps,

$$e_X \colon H^0(L_X) \to k_I,\ e_Y \colon H^0(L_Y) \to k_I,\ f_X \colon H^0(M_X) \to k_J,\ f_Y \colon H^0(M_Y) \to k_J.$$

Let $V := \mathrm{Im}(e_Y)$ and $W := \mathrm{Im}(f_X)$. Let $h_X := \dim V$ and $h_Y := \dim W$. Let

$$G_X := \mathrm{Grass}_{h_X}(k_I) \quad \text{and} \quad G_Y := \mathrm{Grass}_{h_Y}(k_J).$$

Consider the natural actions of the tori $k_I^*$ and $k_J^*$ on $k_I$ and $k_J$, and their respective actions on $G_X$ and $G_Y$. Let $\mathbb{O}_V$ and $\mathbb{O}_W$ denote the orbits of $V$ and $W$ under these actions.

Let $U \subseteq U_\mu$ be an open neighborhood of $\mu$. We divide the proof in three steps.

*Step 1*: We'll show that the closure $\overline{\mathbb{V}}_{\mu,X} \subseteq \mathbb{G}_X$ satisfies

$$\overline{\mathbb{V}}_{\mu,X} = \bigcup_{\overline{\mu} \in U} \mathbb{V}_{\overline{\mu},X}$$

if $g_Y > 0$. In fact, assume that $g_Y > 0$. Then $\alpha \neq 0$. By Lemma 8.4, the map $e_X$ induces a closed embedding $\iota_X \colon G_X \to \mathbb{G}_X$ such that $\iota_X(\mathbb{O}_V) = \mathbb{V}_{\mu,X}$.

For each ordered tripartition $\mathbb{I} = (I', \overline{I}, I'')$ of $I$ let

$$V_{\mathbb{I}} := k_{I'} + (k_{\overline{I}} \cap (V + k_{I''})),$$

and denote by $\mathbb{O}_{V_{\mathbb{I}}}$ the orbit of $V_{\mathbb{I}}$ in $G_X$ under the action of $k_I^*$. By (4.3.1), all the Plücker coordinates of $V$ in $G_X$ are non–zero. Thus, by Lemma 9.1, the closure $\overline{\mathbb{O}}_V \subseteq G_X$ is the union of the orbits $\mathbb{O}_{V_{\mathbb{I}}}$ obtained from all ordered tripartitions $\mathbb{I} = (I', \overline{I}, I'')$ of $I$ such that

$$|I'| < h_X \leq |I - I''|. \tag{10.5.1}$$

Since $h_X = g_Y - |\alpha| + |I|$, Condition (10.5.1) is equivalent to (10.4.1a). By Lemma 10.4, each ordered tripartition $\mathbb{I} = (I', \overline{I}, I'')$ of $I$ satisfying (10.5.1) satisfies (10.4.1c,d) for a certain $\overline{\mu} \in U$. Conversely, for each $\overline{\mu} \in U$ there is an ordered tripartition $\mathbb{I} = (I', \overline{I}, I'')$ of $I$ satisfying (10.4.1c,d) and (10.5.1).

Now, let $\mathbb{I}=(I',\overline{I},I'')$ be an ordered tripartition of $I$ satisfying (10.5.1), and let $\overline{\mu}\in U$ such that (10.4.1c,d) hold. Let

$$\overline{L}_X := L_X(-\textstyle\sum_{p\in I''} x_p) \quad \text{and} \quad \overline{L}_Y := L_Y(-\textstyle\sum_{p\in I'} y_p). \tag{10.5.2}$$

Let $\overline{e}_X$ and $\overline{e}_Y$ be the following compositions,

$$\overline{e}_X\colon H^0(\overline{L}_X) \longrightarrow H^0(L_X) \xrightarrow{e_X} k_I \longrightarrow k_{\overline{I}},$$
$$\overline{e}_Y\colon H^0(\overline{L}_Y) \longrightarrow H^0(L_Y) \xrightarrow{e_Y} k_I \longrightarrow k_{\overline{I}},$$

where in each composition the first map is the natural injection, and the third map is the natural surjection. Equivalently, $\overline{e}_X$ and $\overline{e}_Y$ are the evaluation maps corresponding to the isomorphisms $\overline{\zeta}_{X,p}\colon \overline{L}_X(x_p)\to k$ and $\overline{\zeta}_{Y,p}\colon \overline{L}_Y(y_p)\to k$ induced by $\zeta_{X,p}$ and $\zeta_{Y,p}$ for each $p\in\overline{I}$. Let $\overline{V}:=\mathrm{Im}(\overline{e}_Y)$ and $\overline{h}_X:=h_X-|I'|$. Since $\overline{h}_X>0$ by (10.5.1), from (4.3.1) we get $\dim\overline{V}=\overline{h}_X$. In addition, $V_{\mathbb{I}}=\overline{V}+k_{I'}$. Let $\overline{G}_X:=\mathrm{Grass}_{\overline{h}_X}(k_{\overline{I}})$, and denote by $\overline{\iota}_X\colon\overline{G}_X\to\mathbb{G}_X$ the closed embedding induced by $\overline{e}_X$, as in Lemma 8.4. Let $\mathbb{O}_{\overline{V}}$ denote the orbit of $\overline{V}$ under the natural action of $k^*_{\overline{I}}$. Since $V_{\mathbb{I}}=\overline{V}+k_{I'}$, there is a bijection $\psi_X\colon\mathbb{O}_{\overline{V}}\to\mathbb{O}_{V_{\mathbb{I}}}$ taking a subspace $H\in\mathbb{O}_{\overline{V}}$ to $H+k_{I'}$. Note that

$$\iota_X(\mathbb{O}_{V_{\mathbb{I}}})=\iota_X(\psi_X(\mathbb{O}_{\overline{V}}))=\overline{\iota}_X(\mathbb{O}_{\overline{V}}).$$

By (10.4.1c,d),

$$\overline{L}_X=\omega_X(\textstyle\sum_{p\in\Delta}(1+\alpha_{\overline{\mu},p})x_p) \quad \text{and} \quad \overline{L}_Y=\omega_Y(\textstyle\sum_{p\in I_{\overline{\mu}}}y_p-\sum_{p\in\Delta}\alpha_{\overline{\mu},p}y_p). \tag{10.5.3}$$

So, it follows from Lemma 8.4 that $\overline{\iota}_X(\mathbb{O}_{\overline{V}})=\mathbb{V}_{\overline{\mu},X}$, and hence $\iota_X(\mathbb{O}_{V_{\mathbb{I}}})=\mathbb{V}_{\overline{\mu},X}$. Thus, by Lemma 9.1,

$$\overline{\mathbb{V}}_{\mu,X}=\bigcup_{\overline{\mu}\in U}\mathbb{V}_{\overline{\mu},X},$$

finishing Step 1.

*Step 2*: We'll show that the theorem holds if $g_Xg_Y=0$. In fact, assume that $g_X=0$. By Theorem 8.5,

$$\mathbb{V}_{\mu',Y}=\{H^0(\omega_Y(\textstyle\sum_{p\in\Delta}y_p))\}$$

and $\mathbb{V}_{\mu'}=\mathbb{V}_{\mu',X}\times\mathbb{V}_{\mu',Y}$ for every $\mu'\in\mathbb{Q}^+_\Delta$. If $g_Y=0$ as well, then $\mathbb{V}_{\mu'}=\mathbb{V}_\mu$ for every $\mu'\in\mathbb{Q}^+_\Delta$, and hence the theorem holds. On the other hand, if $g_Y>0$ then the closure $\overline{\mathbb{V}}_\mu\subseteq\mathbb{G}$ satisfies

$$\overline{\mathbb{V}}_\mu=\overline{\mathbb{V}}_{\mu,X}\times\mathbb{V}_{\mu,Y}=\bigcup_{\overline{\mu}\in U}\mathbb{V}_{\overline{\mu},X}\times\mathbb{V}_{\mu,Y}=\bigcup_{\overline{\mu}\in U}\mathbb{V}_{\overline{\mu},X}\times\mathbb{V}_{\overline{\mu},Y}=\bigcup_{\overline{\mu}\in U}\mathbb{V}_{\overline{\mu}},$$

where the second equality holds by Step 1. Now, the case where $g_Y=0$ is analogous. Step 2 is complete.

*Step 3*: We'll show that the theorem holds if $g_Xg_Y>0$. Since $g_X>0$, by Lemma 8.4, the map $f_Y$ induces a closed embedding $\iota_Y\colon G_Y\to\mathbb{G}_Y$ such that $\iota(\mathbb{O}_W)=\mathbb{V}_{\mu,Y}$. Let

$$T:=\{(s,t)\in k^*_I\times k^*_J\,|\,s_p^{\widetilde{\beta}}=t_p^{\widetilde{\alpha}}\text{ for all }p\in I\cap J\}.$$

Let $\mathbb{O}$ be the orbit of $(V,W)$ under the induced action of $T$ on $G_X\times G_Y$. By Lemma 8.4, we have $\iota(\mathbb{O})=\mathbb{V}_\mu$, where $\iota:=\iota_X\times\iota_Y\colon G_X\times G_Y\to\mathbb{G}$.

For each pair of ordered tripartitions $\mathbb{I}=(I',\overline{I},I'')$ of $I$ and $\mathbb{J}=(J',\overline{J},J'')$ of $J$, define the subspaces $V_{\mathbb{I}}\subseteq k_I$ and $W_{\mathbb{J}}\subseteq k_J$ by (9.2.1). By (4.3.1), all the Plücker coordinates of $V$

in $G_X$ and of $W$ in $G_Y$ are non–zero. Thus, by Lemma 9.2, the closure $\overline{\mathbb{O}} \subseteq G_X \times G_Y$ is the union of the orbits under $T$ of the pairs $(V_{\mathbb{I}}, W_{\mathbb{J}})$ obtained from all ordered tripartitions $\mathbb{I} = (I', \overline{I}, I'')$ of $I$ and $\mathbb{J} = (J', \overline{J}, J'')$ of $J$ satisfying (9.2.2,3).

As in Step 1, Condition (9.2.2) is equivalent to Conditions (10.4.1a,b). Since (9.2.3) is equal to (10.4.1g,h), by Lemma 10.4, for each pair consisting of an ordered tripartition $(I', \overline{I}, I'')$ of $I$ and an ordered tripartition $(J', \overline{J}, J'')$ of $J$ satisfying (9.2.2,3), there is a certain $\overline{\mu} \in U$ satisfying (10.4.1c–f). Conversely, for each $\overline{\mu} \in U$ there are ordered tripartitions $(I', \overline{I}, I'')$ of $I$ and $(J', \overline{J}, J'')$ of $J$ satisfying (10.4.1a–h).

Now, let $\mathbb{I} = (I', \overline{I}, I'')$ be an ordered tripartition of $I$ and $\mathbb{J} = (J', \overline{J}, J'')$ an ordered tripartition of $J$ satisfying (9.2.2,3), and let $\overline{\mu} \in U$ such that (10.4.1c–f) hold. Let $\overline{L}_X$ and $\overline{L}_Y$ be given by (10.5.2), and put

$$\overline{M}_X := M_X(-\textstyle\sum_{p\in J'} x_p) \quad \text{and} \quad \overline{M}_Y := M_Y(-\textstyle\sum_{p\in J''} y_p). \tag{10.5.4}$$

Let $\overline{e}_X\colon H^0(\overline{L}_X) \to k_{\overline{I}}$ and $\overline{e}_Y\colon H^0(\overline{L}_Y) \to k_{\overline{I}}$ be the evaluation maps corresponding to the isomorphisms $\overline{\zeta}_{X,p}\colon \overline{L}_X(x_p) \to k$ and $\overline{\zeta}_{Y,p}\colon \overline{L}_Y(y_p) \to k$ induced by $\zeta_{X,p}$ and $\zeta_{Y,p}$ for each $p \in \overline{I}$. Let $\overline{f}_X\colon H^0(\overline{M}_X) \to k_{\overline{J}}$ and $\overline{f}_Y\colon H^0(\overline{M}_Y) \to k_{\overline{J}}$ be the evaluation maps corresponding to the isomorphisms $\overline{\xi}_{X,p}\colon \overline{M}_X(x_p) \to k$ and $\overline{\xi}_{Y,p}\colon \overline{M}_Y(y_p) \to k$ induced by $\xi_{X,p}$ and $\xi_{Y,p}$ for each $p \in \overline{J}$.

Let $\overline{V} := \mathrm{Im}(\overline{e}_Y)$ and $\overline{h}_X := h_X - |I'|$. In addition, let $\overline{G}_X := \mathrm{Grass}_{\overline{h}_X}(k_{\overline{I}})$, and denote by $\overline{\iota}_X\colon \overline{G}_X \to \mathbb{G}_X$ the closed embedding induced by $\overline{e}_X$, as in Lemma 8.4. Let $\overline{W} := \mathrm{Im}(\overline{f}_X)$ and $\overline{h}_Y := h_Y - |J'|$. Then $\dim \overline{W} = \overline{h}_Y$ and $W_{\mathbb{J}} = \overline{W} + k_{J'}$, as in Step 1. Let $\overline{G}_Y := \mathrm{Grass}_{\overline{h}_Y}(k_{\overline{J}})$, and denote by $\overline{\iota}_Y\colon \overline{G}_Y \to \mathbb{G}_Y$ the closed embedding induced by $\overline{f}_Y$. Let $O$ denote the orbit of $(V_{\mathbb{I}}, W_{\mathbb{J}})$ in $G$ under the action of $T$. Let $\mathbb{O}^-$ denote the orbit of $(\overline{V}, \overline{W})$ under the natural action of

$$\overline{T} := \{(s,t) \in k^*_{\overline{I}} \times k^*_{\overline{J}} \,|\, s_p^{\widetilde{\beta}} = t_p^{\widetilde{\alpha}} \text{ for every } p \in \overline{I} \cap \overline{J}\}$$

on $\overline{G} := \overline{G}_X \times \overline{G}_Y$. Since $V_{\mathbb{I}} = \overline{V} + k_{I'}$ and $W_{\mathbb{J}} = \overline{W} + k_{J'}$, there is a bijection $\psi\colon \mathbb{O}^- \to O$ taking a pair $(H_X, H_Y) \in \mathbb{O}^-$ to $(H_X + k_{I'}, H_Y + k_{J'}) \in O$. Let $\overline{\iota} := \overline{\iota}_X \times \overline{\iota}_Y\colon \overline{G}_X \times \overline{G}_Y \to \mathbb{G}$. Note that

$$\iota(O) = \iota(\psi(\mathbb{O}^-)) = \overline{\iota}(\mathbb{O}^-). \tag{10.5.5}$$

By (10.4.1c,d), we have (10.5.3). In addition, by (10.4.1e,f),

$$\overline{M}_X = \omega_X(\textstyle\sum_{p\in J_{\overline{\mu}}} x_p - \sum_{p\in\Delta} \beta_{\overline{\mu},p} x_p) \quad \text{and} \quad \overline{M}_Y = \omega_Y(\sum_{p\in\Delta}(1+\beta_{\overline{\mu},p}) y_p).$$

We will apply Lemma 8.4 to show that $\overline{\iota}(\mathbb{O}^-) = \mathbb{V}_{\overline{\mu}}$. First, note that $\alpha_{\overline{\mu}} \neq 0$. Indeed, $\alpha_{\overline{\mu},p} = \alpha_p$ for every $p \in \overline{I}$ by (10.4.1d). Now, $\overline{I}$ is non–empty and contained in $I$, and $\alpha_p > 0$ for every $p \in I$. Hence $\alpha_{\overline{\mu},p} > 0$ for every $p \in \overline{I}$. Analogously, $\beta_{\overline{\mu}} \neq 0$. So, Lemma 8.4 applies. If $I_{\overline{\mu}} \cap J_{\overline{\mu}} = \emptyset$, we get $\overline{\iota}(\mathbb{O}^-) = \mathbb{V}_{\overline{\mu}}$ immediately.

Assume $I_{\overline{\mu}} \cap J_{\overline{\mu}} \neq \emptyset$, and let $q \in I_{\overline{\mu}} \cap J_{\overline{\mu}}$. Then

$$\widetilde{\alpha}_{\overline{\mu}} = \frac{\alpha_{\overline{\mu},q}}{\gcd(\alpha_{\overline{\mu},q}, \beta_{\overline{\mu},q})} = \frac{\alpha_q}{\gcd(\alpha_q, \beta_q)} = \widetilde{\alpha},$$

where the middle equality follows from (10.4.1c–f). Analogously, $\widetilde{\beta}_{\overline{\mu}} = \widetilde{\beta}$. Since

$$\{(\zeta_{X,p}^{\otimes\widetilde{\beta}} \otimes \xi_{X,p}^{\otimes\widetilde{\alpha}}, \zeta_{Y,p}^{\otimes\widetilde{\beta}} \otimes \xi_{Y,p}^{\otimes\widetilde{\alpha}}) \,|\, p \in I \cap J\}$$

patch $L_X^{\otimes\widetilde{\beta}}\otimes M_X^{\otimes\widetilde{\alpha}}$ and $L_Y^{\otimes\widetilde{\beta}}\otimes M_Y^{\otimes\widetilde{\alpha}}$ to the sheaf $K_{I\cap J}$ given by (8.2.2), it follows from (10.5.2) and (10.5.4) that

$$\{(\overline{\zeta}_{X,p}^{\otimes\widetilde{\beta}_{\overline{\mu}}}\otimes\overline{\xi}_{X,p}^{\otimes\widetilde{\alpha}_{\overline{\mu}}},\overline{\zeta}_{Y,p}^{\otimes\widetilde{\beta}_{\overline{\mu}}}\otimes\overline{\xi}_{Y,p}^{\otimes\widetilde{\alpha}_{\overline{\mu}}})\,|\,p\in I_{\overline{\mu}}\cap J_{\overline{\mu}}\}$$

patch $\overline{L}_X^{\otimes\widetilde{\beta}_{\overline{\mu}}}\otimes\overline{M}_X^{\otimes\widetilde{\alpha}_{\overline{\mu}}}$ and $\overline{L}_Y^{\otimes\widetilde{\beta}_{\overline{\mu}}}\otimes\overline{M}_Y^{\otimes\widetilde{\alpha}_{\overline{\mu}}}$ to the sheaf

$$\overline{K}:=\overline{K}_{I\cap J}(-\widetilde{\beta}_{\overline{\mu}}(\textstyle\sum_{p\in I''}x_p+\sum_{p\in I'}y_p)-\widetilde{\alpha}_{\overline{\mu}}(\sum_{p\in J''}y_p+\sum_{p\in J'}x_p)),$$

where $\overline{K}_{I\cap J}$ is the pull–back of $K_{I\cap J}$ to the blowup $\overline{C}$ of $C$ along $\Delta-(I_{\overline{\mu}}\cap J_{\overline{\mu}})$. Using Equation (8.2.2) and (10.4.1c–f), we get

$$\overline{K}\cong\overline{\omega}^{\otimes(\widetilde{\alpha}_{\overline{\mu}}+\widetilde{\beta}_{\overline{\mu}})}\Big(\sum_{p\notin I_{\overline{\mu}}\cap J_{\overline{\mu}}}(\widetilde{\beta}_{\overline{\mu}}\alpha_{\overline{\mu},p}-\widetilde{\alpha}_{\overline{\mu}}\beta_{\overline{\mu},p})(x_p-y_p)-\widetilde{\alpha}_{\overline{\mu}}\sum_{p\notin J_{\overline{\mu}}}x_p-\widetilde{\beta}_{\overline{\mu}}\sum_{p\notin I_{\overline{\mu}}}y_p\Big),$$

where $\overline{\omega}$ is the pull–back to $\overline{C}$ of the dualizing sheaf $\omega$ of $C$. Applying Lemma 8.4, we get $\overline{\iota}(\mathbb{O}^-)=\mathbb{V}_{\overline{\mu}}$.

Since $\overline{\iota}(\mathbb{O}^-)=\mathbb{V}_{\overline{\mu}}$, we get $\iota(O)=\mathbb{V}_{\overline{\mu}}$ from (10.5.5). Thus, by Lemma 9.2,

$$\overline{\mathbb{V}}_\mu=\bigcup_{\overline{\mu}\in U}\mathbb{V}_{\overline{\mu}},$$

finishing Step 3. □

**Theorem 10.6.** *Preserve* 4.1 *and assume* (4.3.1). *Then the variety $\mathbb{V}$ of limit canonical systems of $C$ is projective and connected. In addition, $\mathbb{V}$ is of pure dimension $\delta-1$, unless $g_X=g_Y=0$; in the exceptional case,*

$$\mathbb{V}=\{(H^0(\omega_X(\textstyle\sum_{p\in\Delta}x_p)),H^0(\omega_Y(\sum_{p\in\Delta}y_p)))\}.\tag{10.6.1}$$

*Proof.* By Proposition 10.3, the covering $\{\mathbb{V}_\mu\mid\mu\in\mathbb{Q}_\Delta^+\}$ of $\mathbb{V}$ is finite. Hence, it follows from Theorem 10.5 that $\mathbb{V}$ is projective.

Let's prove that $\mathbb{V}$ is connected. Let $\mu_1,\mu_2\in\mathbb{Q}_\Delta^+$. For each $t\in[0,1]\cap\mathbb{Q}$ let

$$\mu(t):=(1-t)\mu_1+t\mu_2\quad\text{and}\quad I_t:=\{s\in[0,1]\cap\mathbb{Q}\,|\,\mathbb{V}_{\mu(s)}=\mathbb{V}_{\mu(t)}\}$$

By Proposition 10.3, each $I_t$ is an interval, and finitely many $t_1,\dots,t_n\in[0,1]\cap\mathbb{Q}$ suffice to get a covering $I_{t_1},\dots,I_{t_n}$ of $[0,1]\cap\mathbb{Q}$. Assume $t_1<\cdots<t_n$. Then $\mathbb{V}_{\mu(t_1)}=\mathbb{V}_{\mu_1}$ and $\mathbb{V}_{\mu(t_n)}=\mathbb{V}_{\mu_2}$. By Theorem 10.5, for each $i=1,\dots,n-1$ either $\mathbb{V}_{\mu(t_i)}\subseteq\overline{\mathbb{V}}_{\mu(t_{i+1})}$ or $\overline{\mathbb{V}}_{\mu(t_i)}\supseteq\mathbb{V}_{\mu(t_{i+1})}$; in either case $\overline{\mathbb{V}}_{\mu(t_i)}\cap\overline{\mathbb{V}}_{\mu(t_{i+1})}\neq\emptyset$. Since each $\mathbb{V}_{\mu(t)}$ is irreducible by Theorem 8.5, it follows that $\mathbb{V}_{\mu_1}$ and $\mathbb{V}_{\mu_2}$ lie in the same connected component of $\mathbb{V}$. So $\mathbb{V}$ is connected.

If $g_X=g_Y=0$, then $\alpha_\mu=\beta_\mu=0$ for each $\mu\in\mathbb{Q}_\Delta^+$. Hence (10.6.1) follows from Theorem 8.5.

Assume now that either $g_X>0$ or $g_Y>0$. Let's show that $\mathbb{V}$ is of pure dimension $\delta-1$. Let $\mu\in\mathbb{Z}_\Delta^+$. Then $\dim\mathbb{V}_\mu\leq\delta-1$ by Theorem 8.5. So, it's enough to show that either $\dim\mathbb{V}_\mu=\delta-1$ or there is $\mu'\in\mathbb{Q}_\Delta^+$ such that $\mathbb{V}_\mu\subseteq\overline{\mathbb{V}}_{\mu'}-\mathbb{V}_{\mu'}$. To ease notation, let

$$(\alpha,\rho,I,\beta,\sigma,J,\epsilon):=(\alpha_\mu,\rho_\mu,I_\mu,\beta_\mu,\sigma_\mu,J_\mu,\epsilon_\mu).$$

There are two cases to consider.

*Case 1*: Assume that there is $p \in \Delta$ such that

$$p \notin I \text{ if } |\alpha| > g_Y \text{ and } p \notin J \text{ if } |\beta| > g_X. \tag{10.6.2}$$

Let $p \in \Delta$ satisfying (10.6.2), and put

$$t' := \min\Big(\frac{\rho_p}{1+\alpha_p}, \frac{\mu_p - \sigma_p}{1+\beta_p}\Big).$$

Then $t' > 0$. For each $t \in \mathbb{Q}$ such that $0 \leq t \leq t'$ let $\mu(t) := \mu - tp$. We claim that $t' < \mu_p$. Indeed, if $p \notin I \cap J$ then either $\rho_p < \mu_p$ or $\sigma_p > 0$, and hence $t' < \mu_p$. On the other hand, since $g_X > 0$ or $g_Y > 0$, if $p \in I \cap J$ then $\alpha_p > 0$ or $\beta_p > 0$, and hence $t' \leq \mu_p/2$. Therefore $\mu(t) \in \mathbb{Q}^+_\Delta$ if $0 \leq t \leq t'$.

Now, if $0 \leq t < t'$ then $(\alpha_{\mu(t)}, \beta_{\mu(t)}) = (\alpha, \beta)$. Moreover, it follows from (10.6.2) that $I_{\mu(t)} = I$ if $|\alpha| > g_Y$ and $J_{\mu(t)} = J$ if $|\beta| > g_X$. By Theorem 10.1, $\mathbb{V}_{\mu(t)} = \mathbb{V}_\mu$ if $0 \leq t < t'$. Thus $\mathbb{V}_\mu \subseteq \overline{\mathbb{V}}_{\mu'}$ by Theorem 10.5, where $\mu' := \mu(t')$. On the other hand, either $\alpha_{\mu',p} = \alpha_p + 1$ or $\beta_{\mu',p} = \beta_p + 1$. By Theorem 10.1, $\mathbb{V}_{\mu'} \cap \mathbb{V}_\mu = \emptyset$. So $\mathbb{V}_\mu \subseteq \overline{\mathbb{V}}_{\mu'} - \mathbb{V}_{\mu'}$, finishing the proof.

*Case 2*: Assume now that there is no $p \in \Delta$ satisfying (10.6.2). In addition, assume that $\dim \mathbb{V}_\mu \neq \delta - 1$. We claim that

$$|\alpha| > g_Y, \quad |\beta| > g_X, \quad I \cup J = \Delta \quad \text{and} \quad I \cap J = \emptyset. \tag{10.6.3}$$

Indeed, suppose that $|\alpha| = g_Y$. Since there is no $p \in \Delta$ satisfying (10.6.2), we must have $|\beta| > g_X$. Then $\dim \mathbb{V}_\mu = |J| - 1$ by Theorem 8.5. So $J \neq \Delta$ because $\dim \mathbb{V}_\mu < \delta - 1$. Now, (10.6.2) holds for $p \in \Delta - J$, reaching a contradiction. Thus $|\alpha| > g_Y$. Analogously, $|\beta| > g_X$. Now, (10.6.2) holds for $p \in \Delta - (I \cup J)$. Thus $I \cup J = \Delta$. Finally, if $I \cap J \neq \emptyset$ then $\dim \mathbb{V}_\mu = \delta - 1$ by Theorem 8.5. Thus $I \cap J = \emptyset$, and (10.6.3) holds.

Define $\lambda \in \mathbb{Q}_\Delta$ by letting $\lambda_p := \mu_p$ if $p \in J$ and $\lambda_p := 0$ otherwise. Put

$$t' := \min\Big(\min\Big(\frac{\sigma_p}{\epsilon}\,\Big|\, p \notin J\Big), \min\Big(\frac{\rho_p}{\mu_p(1+\alpha_p)}\,\Big|\, p \in J\Big)\Big).$$

Then $t' > 0$. Let $\mu(t) := \mu - t\lambda$ for each $t \in \mathbb{Q}$ such that $0 \leq t \leq t'$. Now, $\rho_p < \mu_p$ for every $p \in J$ because $I \cap J = \emptyset$. So $t' < 1$, and hence $\mu(t) \in \mathbb{Q}^+_\Delta$ if $0 \leq t \leq t'$. Let $\mu' := \mu(t')$. Then either $\alpha_{\mu',p} = \alpha_{\mu,p} + 1$ for some $p \in J$ or $J_{\mu'} \supset J$. Since $|\beta| > g_X$, in either case $\mathbb{V}_\mu \cap \mathbb{V}_{\mu'} = \emptyset$ by Theorem 10.1. On the other hand,

$$(\alpha_{\mu(t)}, \beta_{\mu(t)}, I_{\mu(t)}, J_{\mu(t)}) = (\alpha, \beta, I, J)$$

if $0 \leq t < t'$. Hence $\mathbb{V}_\mu \subseteq \overline{\mathbb{V}}_{\mu'} - \mathbb{V}_{\mu'}$ as in Case 1. □

**Theorem 10.7.** *Preserve* 4.1 *and assume* (4.3.1). *For each* $\nu = (V_1, V_2) \in \mathbb{G}$, *let*

$$W_\nu := R_{\nu,X} + R_{\nu,Y} + \sum_{p \in \Delta} g(\delta - 2)p,$$

*where* $R_{\nu,X}$ *and* $R_{\nu,Y}$ *are the ramification divisors of the linear systems* $(V_1, \mathbb{L})$ *and* $(V_2, \mathbb{M})$, *respectively. If* $\mathbb{V}$ *is the variety of limit canonical systems of* $C$, *then* $\{W_\nu \,|\, \nu \in \mathbb{V}\}$ *is the set of limit Weierstrass divisors of smoothings of* $C$.

*Proof.* Let $\mu \in \mathbb{Z}^+_\Delta$, and denote by $\widetilde{C}$ the curve gotten from $C$ by splitting the branches of $C$ at each $p \in \Delta$ and connecting them by a chain of $\mu_p - 1$ rational smooth curves; see 4.2. Let

$$L_X := \omega_X(\textstyle\sum_{p\in\Delta}(1+\alpha_{\mu,p})x_p) \quad \text{and} \quad M_Y := \omega_Y(\textstyle\sum_{p\in\Delta}(1+\beta_{\mu,p})y_p). \tag{10.7.1}$$

Let $\nu := (V_1, V_2) \in \mathbb{V}_\mu$. By definition, $(V_1, L_X)$ and $(V_2, M_Y)$ are the limit canonical aspects with foci on $X$ and $Y$ of a certain regular smoothing $\widetilde{\pi}$ of $\widetilde{C}$. Let $\pi$ be the induced smoothing of $C$; see 2.7. By Theorem 5.2, the limit Weierstrass scheme $W$ of $\pi$ satisfies

$$[W] = R_1 + R_2 + \sum_{p\in\Delta} g(g-1-\alpha_{\mu,p}-\beta_{\mu,p})p, \tag{10.7.2}$$

where $R_1$ and $R_2$ are the ramification divisors of $(V_1, L_X)$ and $(V_2, M_Y)$. Now, since

$$\mathbb{L} = L_X(\textstyle\sum_{p\in\Delta}(g_Y - \alpha_{\mu,p})x_p) \quad \text{and} \quad \mathbb{M} = M_Y(\textstyle\sum_{p\in\Delta}(g_X - \beta_{\mu,p})y_p),$$

we have

$$R_1 = R_{\nu,X} - g\textstyle\sum_{p\in\Delta}(g_Y - \alpha_{\mu,p})p \quad \text{and} \quad R_2 = R_{\nu,Y} - g\textstyle\sum_{p\in\Delta}(g_X - \beta_{\mu,p})p.$$

Using the above expressions in (10.7.2), since $g = g_X + g_Y + \delta - 1$, we get $[W] = W_\nu$. So $W_\nu$ is a limit Weierstrass divisor.

Conversely, if $\pi$ is a smoothing of $C$, then there is $\mu \in \mathbb{Z}^+_\Delta$ such that $\pi$ induces a regular smoothing $\widetilde{\pi}$ of the curve $\widetilde{C}$ obtained from $C$ by splitting the branches of $C$ at each $p \in \Delta$ and connecting them by a chain of $\mu_p - 1$ rational smooth curves; see 2.7 and 4.2. Let $L_X$ and $M_Y$ be as in (10.7.1). By Theorem 5.2 there are vector subspaces $V_1 \subseteq H^0(L_X)$ and $V_2 \subseteq H^0(M_Y)$ such that $(V_1, L_X)$ and $(V_2, M_Y)$ are the limit canonical aspects of $\widetilde{\pi}$ with foci on $X$ and $Y$, respectively. Let $\nu := (V_1, V_2) \in \mathbb{G}$. By definition, $\nu \in \mathbb{V}_\mu$. Now, proceed as in the above paragraph to conclude that the limit Weierstrass scheme $W$ of $\pi$ satisfies $[W] = W_\nu$. $\square$

## 11. Miscellany

**Proposition 11.1.** *Preserve* 4.1 *and assume* (4.3.1). *If $g_X > 0$ or $g_Y > 0$ then there is a $(\delta-1)$–dimensional family of limit Weierstrass divisors on $C$.*

*Proof.* Assume that $g_Y > 0$. Then we may choose $\alpha \in \mathbb{Z}^+_\Delta$ such that $|\alpha| = g_Y + \delta - 1$. Define $\mu \in \mathbb{Q}^+_\Delta$ by letting $\mu_p := 1/\alpha_p$ for each $p \in \Delta$. Then $\alpha_\mu = \alpha$ and $I_\mu = \Delta$. Let

$$L_X := \omega_X(\textstyle\sum_{p\in\Delta}(1+\alpha_p)x_p) \quad \text{and} \quad \overline{L}_X := L_X(-\textstyle\sum_{p\in\Delta}x_p).$$

Then $h^0(L_X) = g + \delta - 1$ and $h^0(\overline{L}_X) = g - 1$. For each $q \in X$ let

$$V_q := H^0(L_X(-gq)) + H^0(\overline{L}_X) \subseteq H^0(L_X).$$

Let $U \subseteq X$ be the dense open subset of points $q \in X$ such that $h^0(L_X(-gq)) = \delta - 1$ and $h^0(\overline{L}_X(-gq)) = 0$. Then $\dim V_q = h^0(L_X) - 1$ for each $q \in U$.

We claim that there are $q_1, \dots, q_{\delta-1} \in U$ such that

$$\dim(V_{q_1} \cap \cdots \cap V_{q_j}) = h^0(L_X) - j, \tag{11.1.1a}$$

$$V_{q_1} \cap \cdots \cap V_{q_j} \not\subseteq H^0(L_X(-x_p)) \text{ for any } p \in \Delta, \tag{11.1.1b}$$

for $j = 0, \dots, \delta - 1$. Indeed, let's prove the claim by induction on $j$. First, (11.1.1a,b) hold trivially for $j = 0$. Now, let $j \in \{0, \dots, \delta - 2\}$ and suppose that there are $q_1, \dots, q_j \in U$ such that (11.1.1a,b) hold. Let $V_j := V_{q_1} \cap \cdots \cap V_{q_j}$. Then there is $q \in U$ such that

$$\dim(V_j \cap H^0(L_X(-x_p - gq))) = \delta - j - 2 \quad \text{for every } p \in \Delta. \tag{11.1.2}$$

By (11.1.1a) and (11.1.2),

$$h^0(L_X(-x_p - gq)) = \delta - 2 \quad \text{and} \quad V_j + H^0(L_X(-x_p - gq)) = H^0(L_X)$$

for each $p \in \Delta$. Now, if $V_j \subseteq V_q$ then $V_j + H^0(L_X(-gq)) \subseteq V_q$, and hence $V_q = H^0(L_X)$. However, $\dim V_q = h^0(L_X) - 1$ because $q \in U$, reaching a contradiction. So $V_j \not\subseteq V_q$, and hence $\dim(V_j \cap V_q) = h^0(L_X) - j - 1$.

In addition, if $V_j \cap V_q \subseteq H^0(L_X(-x_p))$ for a certain $p \in \Delta$, then

$$V_j \cap V_q = V_j \cap (H^0(L_X(-x_p - gq)) + H^0(\overline{L}_X)).$$

Now, $h^0(\overline{L}_X(-gq)) = 0$ because $q \in U$. Moreover, $V_j \supseteq H^0(\overline{L}_X)$. So

$$\dim(V_j \cap H^0(L_X(-x_p - gq))) = \dim(V_j \cap V_q) - h^0(\overline{L}_X).$$

Since $\dim(V_j \cap V_q) = h^0(L_X) - j - 1$, it follows that

$$\dim(V_j \cap H^0(L_X(-x_p - gq))) = h^0(L_X) - j - 1 - (g - 1) = \delta - j - 1,$$

contradicting (11.1.2). Hence $V_j \cap V_q \not\subseteq H^0(L_X(-x_p))$ for any $p \in \Delta$. The induction proof of our claim is complete.

For each $D = (q_1, \dots, q_{\delta-1}) \in U^{\delta-1}$ let $V_D := V_{q_1} \cap \cdots \cap V_{q_{\delta-1}}$. The following conditions on $D \in U^{\delta-1}$,

(11.1.3a) $$\dim V_D = g,$$

(11.1.3b) $$V_D \not\subseteq H^0(L_X(-x_p)) \text{ for any } p \in \Delta,$$

define an open subset $W \subseteq U^{\delta-1}$. By our claim, $W \neq \emptyset$.

Let $D := (q_1, \dots, q_{\delta-1}) \in W$. Since $V_{q_j} \supseteq H^0(\overline{L}_X)$ for each $j = 1, \dots, \delta - 1$, also $V_D \supseteq H^0(\overline{L}_X)$. As shown in 8.6, Conditions (11.1.3) imply that $V_D \in \mathbb{V}_{\mu,X}$. Since $V_D + H^0(L_X(-gq_j)) \subseteq V_{q_j}$, and since $\dim V_{q_j} = g + \delta - 2$ and $h^0(L_X(-gq_j)) = \delta - 1$, it follows that $V_D \cap H^0(L_X(-gq_j)) \neq 0$ for each $j = 1, \dots, \delta - 1$. So $q_1, \dots, q_{\delta-1}$ are ramification points of $(V_D, L_X)$, and hence of $(V_D, \mathbb{L})$. Since $V_D \in \mathbb{V}_{\mu,X}$ it follows from Theorem 10.7 that $q_1, \dots, q_{\delta-1}$ sit on the support of a limit Weierstrass divisor on $C$. Since $\dim W = \delta - 1$, there must be a $(\delta - 1)$–dimensional family of limit Weierstrass divisors. $\square$

**Theorem 11.2.** *Preserve* 4.1 *and assume* (4.3.1). *Let $\mathbb{V}$ be the variety of limit canonical systems of $C$, and denote by $N(\mathbb{V})$ the number of its irreducible components. Set*

$$n_\delta(h) := \binom{h + \delta - 1}{\delta} - \binom{h}{\delta} \quad \text{and} \quad g_{i,j} := \gcd(g_X + i, g_Y + j)$$

*for all $h, i, j \in \mathbb{Z}^+$. Assume that $\delta > 1$. Then the following two statements hold.*

1. *If $g_X g_Y = 0$ or $g_X = g_Y$ then $N(\mathbb{V}) = n_\delta(\max(g_X, g_Y))$, unless $g_X = g_Y = 0$; in the exceptional case, $N(\mathbb{V}) = 1$.*
2. *If $g_X g_Y > 0$ then*

$$N(\mathbb{V}) \geq n_\delta(g_X) + n_\delta(g_Y) - \sum_{i,j=1}^{\delta-1} \binom{g_{i,j} - 1}{\delta - 1}.$$

3. *$\mathbb{V}$ is irreducible if and only if $g_X \leq 1$ and $g_Y \leq 1$.*

*Proof.* If $g_X = g_Y = 0$ then $\mathbb{V}$ is simply a point by Theorem 10.6, thus proving Statements 1 and 3 in this case. We may assume from now on that $g_Y > 0$. (The case where $g_X > 0$ is completely analogous.)

Given $i \in \mathbb{Z}^+$, the number of $\alpha \in \mathbb{Z}^+_\Delta$ such $|\alpha| = i$ is $\binom{i-1}{\delta-1}$. Thus the number of $\alpha \in \mathbb{Z}^+_\Delta$ such that $i < |\alpha| < i + \delta$ is $n_\delta(i)$.

If $h \in \mathbb{Z}^+$ then $n_\delta(h) = 1$ if and only if $h = 1$. Indeed, $n_\delta(1) = 1$ because $\delta > 1$. In addition, if $h > 1$ then the number of $\alpha \in \mathbb{Z}^+_\Delta$ such that $|\alpha| = h + \delta - 1$ is at least $\delta$, and hence $n_\delta(h) > 1$ because $\delta > 1$.

Let $N_X(\mathbb{V})$ be the number of strata $\mathbb{V}_\mu$ of $\mathbb{V}$ with $\dim \mathbb{V}_{\mu,X} = \delta - 1$. By Theorem 8.5, $\dim \mathbb{V}_{\mu,X} = \delta - 1$ if and only if $I_\mu = \Delta$ and $|\alpha_\mu| > g_Y$. Now, given $\alpha \in \mathbb{Z}^+_\Delta$ satisfying $g_Y < |\alpha| < g_Y + \delta$, there is $\mu \in \mathbb{Q}^+_\Delta$ such that $I_\mu = \Delta$ and $\alpha_\mu = \alpha$. (For instance, pick $\mu \in \mathbb{Q}^+_\Delta$ given by $\mu_p := 1/\alpha_p$ for each $p \in \Delta$.) If also $\mu' \in \mathbb{Q}^+_\Delta$ satisfies $I_{\mu'} = \Delta$ and $\alpha_{\mu'} = \alpha$ then there is $t \in \mathbb{Q}^+$ such that $\mu' = t\mu$, and hence $\mathbb{V}_{\mu'} = \mathbb{V}_\mu$ by Theorem 10.1. It follows that sending $\mathbb{V}_\mu$ to $\alpha_\mu$ gives a 1–1 correspondence between the set of strata $\mathbb{V}_\mu$ of $\mathbb{V}$ with $\dim \mathbb{V}_{\mu,X} = \delta - 1$ and the set of $\alpha \in \mathbb{Z}^+_\Delta$ with $g_Y < |\alpha| < g_Y + \delta$. Thus $N_X(\mathbb{V}) = n_\delta(g_Y)$.

If $g_X = 0$ then $N(\mathbb{V}) = N_X(\mathbb{V})$, thus proving Statement 1 in this case. Moreover, $N(\mathbb{V}) = 1$ if and only if $g_Y = 1$, thus proving Statement 3 in this case.

Assume from now on that $g_X > 0$. By analogy, $N_Y(\mathbb{V}) = n_\delta(g_X)$, where $N_Y(\mathbb{V})$ is the number of strata $\mathbb{V}_\mu$ of $\mathbb{V}$ with $\dim \mathbb{V}_{\mu,Y} = \delta - 1$.

Fix $i, j \in \{1, \dots, \delta - 1\}$. Let $\alpha, \beta \in \mathbb{Z}^+_\Delta$ such that

$$|\alpha| = g_Y + i \quad \text{and} \quad |\beta| = g_X + j. \tag{11.2.1}$$

Then there is $\mu \in \mathbb{Q}^+_\Delta$ such that

$$\mu_p\alpha_p = \mu_q\alpha_q \quad \text{and} \quad \mu_p\beta_p = \mu_q\beta_q \quad \text{for all } p, q \in \Delta \tag{11.2.2}$$

if and only if

$$\frac{\alpha_p}{\beta_p} = \frac{g_Y + i}{g_X + j} \quad \text{for every } p \in \Delta. \tag{11.2.3}$$

Now, $\alpha, \beta \in \mathbb{Z}^+_\Delta$ satisfy (11.2.1,3) if and only if there is $\tau \in \mathbb{Z}^+_\Delta$ such that

$$|\tau| = g_{i,j}, \quad \alpha = \frac{g_Y + i}{g_{i,j}}\tau, \quad \beta = \frac{g_X + j}{g_{i,j}}\tau.$$

So, there is a 1–1 correspondence between the set of pairs $(\alpha, \beta) \in \mathbb{Z}^+_\Delta \times \mathbb{Z}^+_\Delta$ satisfying (11.2.1) and (11.2.2) for a certain $\mu \in \mathbb{Q}^+_\Delta$ and the set of $\tau \in \mathbb{Z}^+_\Delta$ such that $|\tau| = g_{i,j}$. The number of $\tau \in \mathbb{Z}^+_\Delta$ such that $|\tau| = g_{i,j}$ is $\binom{g_{i,j}-1}{\delta-1}$.

Let $N$ be the number of strata $\mathbb{V}_\mu$ of $\mathbb{V}$ such that

$$\dim \mathbb{V}_{\mu,X} = \dim \mathbb{V}_{\mu,Y} = \delta - 1. \tag{11.2.4}$$

Then $N(\mathbb{V}) \geq N_X(\mathbb{V}) + N_Y(\mathbb{V}) - N$. Now, by Theorem 8.5, Condition (11.2.4) holds if and only if

$$I_\mu = J_\mu = \Delta, \quad |\alpha_\mu| > g_Y, \quad |\beta_\mu| > g_X.$$

It follows from Theorem 10.1 that sending $\mathbb{V}_\mu$ to $(\alpha_\mu, \beta_\mu)$ gives a 1–1 correspondence between the set of strata $\mathbb{V}_\mu$ of $\mathbb{V}$ satisfying (11.2.4) and the set of pairs $(\alpha, \beta) \in \mathbb{Z}^+_\Delta \times \mathbb{Z}^+_\Delta$ satisfying (11.2.1) for certain $i, j \in \{1, \dots, \delta - 1\}$ and (11.2.2) for a certain $\mu \in \mathbb{Q}^+_\Delta$. Thus

$$N = \sum_{i,j=1}^{\delta-1} \binom{g_{i,j} - 1}{\delta - 1}.$$

Since $N(\mathbb{V}) \geq N_X(\mathbb{V}) + N_Y(\mathbb{V}) - N$, we proved Statement 2.

Without loss of generality, assume from now on that $g_Y \geq g_X$. Clearly $N(\mathbb{V}) \geq N_X(\mathbb{V})$, and hence $N(\mathbb{V}) \geq n_\delta(g_Y)$. So, $N(\mathbb{V}) = 1$ only if $n_\delta(g_Y) = 1$, and hence only if $g_Y = 1$. It follows that $\mathbb{V}$ is irreducible only if $g_X = g_Y = 1$.

Assume from now on that $g_X = g_Y$. Then $(\alpha_\mu, I_\mu) = (\beta_\mu, J_\mu)$ for each $\mu \in \mathbb{Q}^+_\Delta$. It follows from Theorem 8.5 that $\dim \mathbb{V}_{\mu,X} = \delta - 1$ if and only if $\dim \mathbb{V}_\mu = \delta - 1$. Hence $N(\mathbb{V}) = N_X(\mathbb{V})$, thus proving Statement 1. Now, if $g_X = g_Y = 1$ then $N(\mathbb{V}) = 1$ by Statement 1. $\square$

**Example 11.3.** Preserve 4.1 and assume (4.3.1). Assume that $g_X = g_Y = 1$. Then $g = \delta + 1$. Choose a basis for $H^0(\omega^{\otimes 2})$ and consider the corresponding bicanonical map $\phi \colon C \to \mathbb{P}^{3\delta-1}$. Let $H$ (resp. $H_X$, resp. $H_Y$) be the subspace of $\mathbb{P}^{3\delta-1}$ spanned by the image of $\Delta$ (resp. $X$, resp. $Y$) under $\phi$. Then $\dim H = \delta - 1$ and

$$\dim H_X = \dim H_Y = 2\delta - 1.$$

Moreover, $H_X \cap H_Y = H$. We may view each hyperplane $L \subseteq H$ as a subspace of both $H_X$ and $H_Y$, and compose $\phi|_X \colon X \to H_X$ and $\phi|_Y \colon Y \to H_Y$ with the projections from $H_X$ and $H_Y$ centered at $L$. Since $\omega_X$ and $\omega_Y$ are trivial, these compositions correspond to vector subspaces $V_{L,X} \subseteq H^0(\omega_X(2\Delta))$ and $V_{L,Y} \subseteq H^0(\omega_Y(2\Delta))$ of dimension $g$. It follows from the discussion in 8.7 that

$$\mathbb{V} = \{(V_{L,X}, V_{L,Y}) \mid L \text{ is a hyperplane of } H\}.$$

**11.4.** *Graphic representation.* Preserve 4.1 and assume (4.3.1). For each $\lambda \in \mathbb{Z}^+_\Delta$ and each non–empty subset $K \subseteq \Delta$ let

$$\begin{aligned} \mathbb{U}_{X,\lambda,K} :=& \{\mu \in \mathbb{Q}^+_\Delta \mid \alpha_\mu = \lambda \text{ and } I_\mu = K\}, \\ \mathbb{U}_{Y,\lambda,K} :=& \{\mu \in \mathbb{Q}^+_\Delta \mid \beta_\mu = \lambda \text{ and } J_\mu = K\}. \end{aligned}$$

Then $\mathbb{U}_{X,\lambda,K}$ and $\mathbb{U}_{Y,\lambda,K}$ are homogeneous. Now, $\mu \in \mathbb{U}_{X,\lambda,K}$ if and only if Condition (14.4.1) below holds.

$$\text{(11.4.1)}\quad \text{There's } \rho \in \mathbb{Q}^+_\Delta \text{ such that } \begin{cases} 0 < \rho_p \leq \mu_p \text{ for every } p \in \Delta, \\ \rho_p = \mu_p \text{ if and only if } p \in K, \\ \mu_p(\lambda_p + 1) - \rho_p = \mu_q(\lambda_q + 1) - \rho_q \text{ for all } p, q \in \Delta. \end{cases}$$

Condition (11.4.1) can be restated as

$$\text{(11.4.2)}\qquad \frac{\mu_p}{\mu_q} \in \begin{cases} \left(\frac{\lambda_q}{\lambda_p+1}, \frac{\lambda_q}{\lambda_p}\right] & \text{if } q \in K, \\ \left[\frac{\lambda_q}{\lambda_p}, \frac{\lambda_q+1}{\lambda_p}\right) & \text{if } p \in K, \\ \left(\frac{\lambda_q}{\lambda_p+1}, \frac{\lambda_q+1}{\lambda_p}\right) & \text{if } p, q \notin K. \end{cases}$$

Similar conditions define the points $\mu \in \mathbb{U}_{Y,\lambda,K}$. Condition (11.4.2) shows that $\mathbb{U}_{X,\lambda,K}$ is convex. Analogously, $\mathbb{U}_{Y,\lambda,K}$ is convex as well.

For each $\mu \in \mathbb{Q}^+_\Delta$ let

$$\begin{aligned} \mathbb{U}_{X,\mu} :=& \{\mu' \in \mathbb{Q}^+_\Delta \mid \mathbb{V}_{\mu',X} = \mathbb{V}_{\mu,X}\}, \\ \mathbb{U}_{Y,\mu} :=& \{\mu' \in \mathbb{Q}^+_\Delta \mid \mathbb{V}_{\mu',Y} = \mathbb{V}_{\mu,Y}\}, \\ \mathbb{U}_\mu :=& \{\mu' \in \mathbb{Q}^+_\Delta \mid \mathbb{V}_{\mu'} = \mathbb{V}_\mu\}. \end{aligned}$$

Then $\mathbb{U}_\mu = \mathbb{U}_{X,\mu} \cap \mathbb{U}_{Y,\mu}$ by Theorem 10.1. In addition,

$$\mathbb{U}_{X,\mu} = \begin{cases} \mathbb{U}_{X,\alpha_\mu,I_\mu} & \text{if } |\alpha_\mu| > g_Y, \\ \bigcup_{I\subseteq\Delta} \mathbb{U}_{X,\alpha_\mu,I} & \text{if } |\alpha_\mu| = g_Y, \end{cases} \tag{11.4.3a}$$

$$\mathbb{U}_{Y,\mu} = \begin{cases} \mathbb{U}_{Y,\beta_\mu,J_\mu} & \text{if } |\beta_\mu| > g_X, \\ \bigcup_{J\subseteq\Delta} \mathbb{U}_{Y,\beta_\mu,J} & \text{if } |\beta_\mu| = g_X. \end{cases} \tag{11.4.3b}$$

In particular, $\mathbb{U}_\mu$, $\mathbb{U}_{X,\mu}$ and $\mathbb{U}_{Y,\mu}$ are convex and homogeneous.

Using (14.4.2,3) we can represent graphically the covering of $\mathbb{Q}^+_\Delta$ by the strata $\mathbb{U}_\mu$, and then study their interrelations. Since there is a 1–1 correspondence between the $\mathbb{U}_\mu$ and the $\mathbb{V}_\mu$ this graphic representation is useful in the study of the variety of limit canonical systems $\mathbb{V}$. For instance, using Theorem 10.5 and the graphic representation we can determine whether two irreducible components of $\mathbb{V}$ intersect, and what the intersection is. We can also compute the number of irreducible components of $\mathbb{V}$ using Theorem 8.5 and the graphic representation. See 11.5 and 11.6 below.

As the $\mathbb{U}_\mu$ are homogeneous, it's more convenient to fix $q \in \Delta$ and consider the map

$$\mathbb{Q}^+_\Delta \to \mathbb{Q}^+_{\Delta-\{q\}}, \text{ given by } \mu \mapsto \overline{\mu}, \text{ where } \overline{\mu}_p := \mu_p/\mu_q \text{ for each } p \in \Delta - \{q\}. \tag{11.4.4}$$

So, instead of depicting the covering of $\mathbb{Q}^+_\Delta$ by the $\mathbb{U}_\mu$, we can depict the covering of $\mathbb{Q}^+_{\Delta-\{q\}}$ by the corresponding images of the $\mathbb{U}_\mu$.

**11.5.** *Curves with $\delta = 2$.* Preserve 4.1 and assume (4.3.1). Assume $\delta = 2$. By Theorem 8.5, a stratum $\mathbb{V}_\mu$ of the variety of limit canonical systems $\mathbb{V}$ has dimension 1 if and only if either $|\alpha_\mu| = g_Y + 1$ or $|\beta_\mu| = g_X + 1$. Fix $q \in \Delta$, and identify $\mathbb{Q}^+_{\Delta-\{q\}}$ with $\mathbb{Q}^+$. Under the map (11.4.4), the strata $\mathbb{U}_\mu$ are sent to the intervals in $\mathbb{Q}^+$ depicted in Figure 3, where we assume that $g_Y > g_X > 0$.

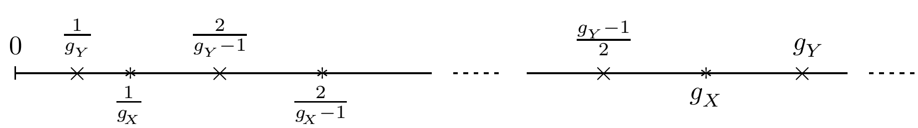


FIGURE 3. Graphic representation for $\delta = 2$.

The points marked as "$\times$" (resp. "$*$") correspond to the strata $\mathbb{V}_\mu$ of $\mathbb{V}$ with $|\alpha_\mu| = g_Y + 1$ (resp. $|\beta_\mu| = g_X + 1$). The number of distinct marked points on the line in Figure 3 is the number $N(\mathbb{V})$ of irreducible components of $\mathbb{V}$. Thus

$$N(\mathbb{V}) = g_X + g_Y - \gcd(g_X + 1, g_Y + 1) + 1.$$

So, if $\delta = 2$ then equality holds in Statement 2 of Theorem 11.2.

The open intervals on the line in Figure 3 correspond to the strata $\mathbb{V}_\mu$ of $\mathbb{V}$ with $|\alpha_\mu| = g_Y$ and $|\beta_\mu| = g_X$, and hence represent points of $\mathbb{V}$. By Theorem 10.5, the stratum of $\mathbb{V}$ corresponding to a certain interval in Figure 3 is the intersection of the closures in $\mathbb{G}$ of the one–dimensional strata of $\mathbb{V}$ corresponding to each end of the interval. Therefore, $\mathbb{V}$ is as described in Figure 4. Each irreducible component of $\mathbb{V}$ is a projective line, unless it corresponds to a point on the line in Figure 3 that is simultaneously marked as "$\times$" and

"$*$". In this case, the irreducible component is isomorphic to a rational curve of bidegree

$$\frac{(g_X+1, g_Y+1)}{\gcd(g_X+1, g_Y+1)}$$

in the quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

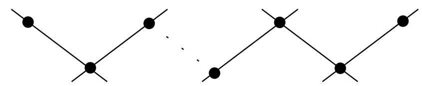

FIGURE 4. The variety of limit canonical systems for $\delta = 2$.

**11.6.** *Curves with* $\delta = 3$. Preserve 4.1 and assume (4.3.1). Assume $\delta = 3$. Fix $r \in \Delta$, and identify $\mathbb{Q}^+_{\Delta-\{r\}}$ with $\mathbb{Q}^+ \times \mathbb{Q}^+$. Under the map (11.4.4), the strata $\mathbb{U}_\mu$ are sent to convex regions in $\mathbb{Q}^+ \times \mathbb{Q}^+$ of dimensions varying from 0 to 2. These dimensions are "dual" to those of the corresponding strata $\mathbb{V}_\mu$ of the variety of limit canonical systems $\mathbb{V}$. More precisely, the image of $\mathbb{U}_\mu$ in $\mathbb{Q}^+ \times \mathbb{Q}^+$ has dimension $i$ if and only if $\dim \mathbb{V}_\mu = 2 - i$.

We depict the decomposition of $\mathbb{Q}^+ \times \mathbb{Q}^+$ for $g_X = g_Y = 3$ in Figure 5a, and for $g_X = 2$ and $g_Y = 4$ in Figure 5b. In Figure 5b the solid lines depict the decomposition of $\mathbb{Q}^+ \times \mathbb{Q}^+$ by the images under (11.4.4) of the strata $\mathbb{U}_{\mu,X}$, whereas the dashed lines depict the decomposition given by the strata $\mathbb{U}_{\mu,Y}$.

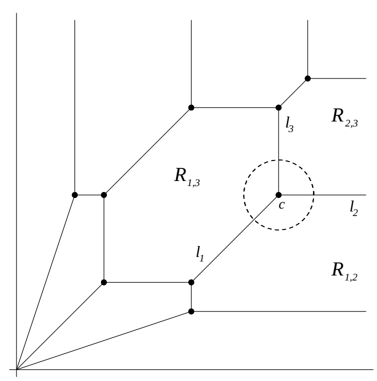


FIGURE 5a: $g_X = g_Y = 3$.

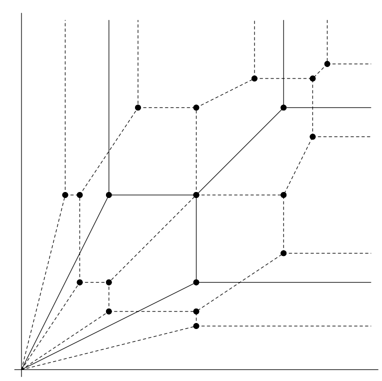

FIGURE 5b: $g_X = 2$ and $g_Y = 4$.

The open disc in Figure 5a corresponds to one of the neighborhoods $U_\mu$ whose existence was claimed by Theorem 10.5. More precisely, let $\mu \in \mathbb{Q}^+_\Delta$ whose image under the map (11.4.4) is the center $c$ of the disc, and let $U_\mu \subseteq \mathbb{Q}^+_\Delta$ be the inverse image of the disc under this same map. Then the closure $\overline{\mathbb{V}}_\mu \subseteq \mathbb{G}$ satisfies

$$\overline{\mathbb{V}}_\mu = \bigcup_{\overline{\mu} \in U_\mu} \mathbb{V}_{\overline{\mu}}.$$

More precisely, $\overline{\mathbb{V}}_\mu$ is an irreducible component of $\mathbb{V}$ and

$$\overline{\mathbb{V}}_\mu - \mathbb{V}_\mu = \overline{\mathbb{V}}_{\mu_1} \cup \overline{\mathbb{V}}_{\mu_2} \cup \overline{\mathbb{V}}_{\mu_3},$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{Q}^+_\Delta$ are (any) points having images under (11.4.4) in the interiors of the three segments of lines $l_1, l_2, l_3$ meeting at $c$, as shown in Figure 5a. Each $\overline{\mathbb{V}}_{\mu_i}$ is irreducible

of dimension 1. In addition, if $1 \leq i < j \leq 3$ then $\overline{\mathbb{V}}_{\mu_i} \cap \overline{\mathbb{V}}_{\mu_j} = \mathbb{V}_{\mu_{i,j}}$ where $\mu_{i,j} \in \mathbb{Q}^+_\Delta$ is (any) point having image under (11.4.4) in the open region $R_{i,j}$ whose boundary contains $l_i$ and $l_j$, as shown in Figure 5a. Each $\mathbb{V}_{\mu_{i,j}}$ is a point.

Let $N(\mathbb{V})$ be the number of irreducible components of $\mathbb{V}$. If $g_X = g_Y = 3$ then $N(\mathbb{V})$ is the number of marked points in Figure 5a. So $N(\mathbb{V}) = 9$, as shown in Theorem 11.2. If $g_X = 2$ and $g_Y = 4$ then $N(\mathbb{V})$ is the number of points in Figure 5b that are either marked or in the intersection of a solid line and a dashed line. There are 19 marked points and 6 points of intersection of a solid line and a dashed line in Figure 5b. Thus $N(\mathbb{V}) = 25$. Note that in this case Theorem 11.2 says simply that $N(\mathbb{V}) \geq 19$.

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